Interval Temporal Logic Proofs

Antonio Cau, Ben Moszkowski and David Smallwood

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HTML version of Interval Temporal Proofs

Contents

Part I

Imperative Reasoning proofs

Proofs taken from

Ben C. Moszkowski. "Imperative Reasoning in Interval Temporal Logic", Internal Report, University of Newcastle upon Tyne.

1 A Proof System

1.1 Propositional Axioms and Inference Rules

The proof system uses a number of the propositional axioms suggested by Rosner and Pnueli but also includes our own axioms and inference rules for the operators $\boxdot$ and *chop-plus*.

$\vdash$	All substitution instances of valid propositional *chop*-free temporal logic formulas for finite time	PTL
$\vdash$	$(f ; g) ; h \equiv f ; (g ; h)$	ChopAssoc
$\vdash$	$(f \vee f_1) ; g \supset (f ; g) \vee (f_1 ; g)$	OrChopImp
$\vdash$	$f ; (g \vee g_1) \supset (f ; g) \vee (f ; g_1)$	ChopOrImp
$\vdash$	$\text{empty} ; f \equiv f$	EmptyChop
$\vdash$	$f ; \text{empty} \equiv f$	ChopEmpty
$\vdash$	$\boxdot(f \supset f_1) \wedge \Box(g \supset g_1) \supset (f ; g) \supset (f_1 ; g_1)$	BiBoxChopImpChop
$\vdash$	$w \supset \boxdot w$	StateImpBi
$\vdash$	$\bigcirc f \supset \neg \bigcirc \neg f$	NextImpNotNextNot
$\vdash$	$\Box(f \supset ⓦ f) \wedge f \supset \Box f$	BoxInduct
$\vdash$	$f^{+} \equiv f \vee (f \wedge \text{more}) ; f^{+}$	ChopPlusEqv
$\vdash$	$f \supset g, \quad \vdash \quad f \quad \Rightarrow \quad \vdash \quad g$	MP
$\vdash$	$f \quad \Rightarrow \quad \vdash \quad \Box f$	BoxGen
$\vdash$	$f \quad \Rightarrow \quad \vdash \quad \boxdot f$	BiGen

Instead of **ChopPlusEqv** one can use

$\vdash$	$f^{*} \equiv \text{empty} \vee (f \wedge \text{more}) ; f^{*}$	ChopStarEqv

1.2 Propositional proofs

Before we prove some theorems, it is worth mentioning a few useful theorems and derived rules. They are all solely based on propositional logic and temporal logic without *chop* and we omit their proofs.

$\vdash f_1 \supset f_2, \quad ,\ldots, \quad \vdash f_{n-1} \supset f_n \Rightarrow \quad \vdash f_1 \supset f_n$		ImpChain
$\vdash f_1 \equiv f_2, \quad ,\ldots, \quad \vdash f_{n-1} \equiv f_n \Rightarrow \quad \vdash f_1 \equiv f_n$		EqvChain
$\vdash f_1, \vdash f_2, \ldots, \vdash f_n \quad \Rightarrow \quad \vdash g,$		Prop
where the formula $f_1 \wedge f_2 \wedge \ldots f_n \supset g$		
is a substitution instance of a propositional tautology		

2 Propositional Interval Temporal Logic Theorems

2.1 Basic ITL Theorems

$\vdash \quad (f \wedge f_1) ; g \supset f ; g$ AndChopA

Proof:

1	$\vdash$	$f \wedge f_1 \supset f$	**PTL**
2	$\vdash$	$\boxdot(f \wedge f_1 \supset f)$	1, **BiGen**
3	$\vdash$	$\boxdot(f \wedge f_1 \supset f) \supset (f \wedge f_1) ; g \supset f ; g$	2, **BiChopImpChop**
4	$\vdash$	$(f \wedge f_1) ; g \supset f ; g$	2, 3, **MP**

qed

The following related theorem has a similar proof:

$\vdash \quad (f \wedge f_1) ; g \supset f_1 ; g$ AndChopB

Proof:

1	$\vdash$	$f \wedge f_1 \supset f_1$	**PTL**
2	$\vdash$	$\boxdot(f \wedge f_1 \supset f_1)$	1, **BiGen**
3	$\vdash$	$\boxdot(f \wedge f_1 \supset f_1) \supset (f \wedge f_1) ; g \supset f_1 ; g$	2, **BiChopImpChop**
4	$\vdash$	$(f \wedge f_1) ; g \supset f_1 ; g$	2, 3, **MP**

qed

$\vdash \quad (\bigcirc f) ; g \equiv \bigcirc(f ; g)$ NextChop

Proof:

1	$\vdash$	$(\mathrm{skip} ; f) ; g \equiv \mathrm{skip} ; (f ; g)$	**ChopAssoc**
2	$\vdash$	$(\bigcirc f) ; g \equiv \bigcirc(f ; g)$	1, def. of $\bigcirc$

qed

$\vdash \quad \boxdot(f \supset f_1) \supset (f ; g) \supset (f_1 ; g)$ BiChopImpChop

Proof:

1	$\vdash$	$g \supset g$	**Prop**
2	$\vdash$	$\Box(g \supset g)$	1, **BoxGen**
3	$\vdash$	$\boxdot(f \supset f_1) \wedge \Box(g \supset g) \supset (f ; g) \supset (f_1 ; g)$	**BiBoxChopImpChop**
4	$\vdash$	$\boxdot(f \supset f_1) \supset (f ; g) \supset (f_1 ; g)$	2, 3, **Prop**

qed

$$\vdash \quad \Box(g \supset g_1) \supset (f ; g) \supset (f ; g_1)$$ BoxChopImpChop

Proof:

1	$\vdash$	$f \supset f$	**Prop**
2	$\vdash$	$\boxdot(f \supset f)$	1, **BiGen**
3	$\vdash$	$\boxdot(f \supset f) \wedge \Box(g \supset g_1) \supset (f ; g) \supset (f ; g_1)$	**BiBoxChopImpChop**
4	$\vdash$	$\Box(g \supset g_1) \supset (f ; g) \supset (f ; g_1)$	2, 3, **Prop**

qed

$$\vdash f \supset f_1 \quad \Rightarrow \quad \vdash f ; g \supset f_1 ; g$$ LeftChopImpChop

Proof:

1	$\vdash$	$f \supset f_1$	given
2	$\vdash$	$\boxdot(f \supset f_1)$	1, **BiGen**
3	$\vdash$	$\boxdot(f \supset f_1) \supset f ; g \supset f_1 ; g$	**BiChopImpChop**
4	$\vdash$	$f ; g \supset f_1 ; g$	2, 3, **MP**

qed

$$\vdash g \supset g_1 \quad \Rightarrow \quad \vdash f ; g \supset f ; g_1$$ RightChopImpChop

Proof:

1	$\vdash$	$g \supset g_1$	given
2	$\vdash$	$\Box(g \supset g_1)$	**BoxGen**
3	$\vdash$	$\Box(g \supset g_1) \supset (f ; g) \supset (f ; g_1)$	**BoxChopImpChop**
4	$\vdash$	$f ; g \supset f ; g_1$	2, 3, **MP**

qed

Here is a derived rule that is a corollary of **RightChopImpChop**:

$$\vdash g \equiv g_1 \quad \Rightarrow \quad \vdash f ; g \equiv f ; g_1$$ RightChopEqvChop

Proof:

1	$\vdash$	$g \equiv g_1$	given
2		$\vdash g \supset g_1 \quad \Rightarrow \quad \vdash f ; g \supset f ; g_1$	RightChopImpChop
3		$\vdash g_1 \supset g \quad \Rightarrow \quad \vdash f ; g_1 \supset f ; g$	RightChopImpChop
4		$\vdash g \equiv g_1 \quad \Rightarrow \quad \vdash f ; g \equiv f ; g_1$	2, 3

qed

$$\vdash \quad f ; (g \lor g_1) \equiv f ; g \lor f ; g_1 \qquad \text{ChopOrEqv}$$

The proof for $\subset$ is immediate from axiom **ChopOrImp**.

Here is the proof for the converse:

1	$\vdash$	$g \supset g \lor g_1$	**Prop**
2	$\vdash$	$f ; g \supset f ; (g \lor g_1)$	1, **RightChopImpChop**
3	$\vdash$	$g_1 \supset g \lor g_1$	**Prop**
4	$\vdash$	$f ; g_1 \supset f ; (g \lor g_1)$	3, **RightChopImpChop**
5	$\vdash$	$f ; g \lor f ; g_1 \supset f ; (g \lor g_1)$	2, 4, **Prop**

qed

$$\vdash f \supset f_1 \lor f_2 \quad \Rightarrow \quad \vdash f ; g \supset (f_1 ; g) \lor (f_2 ; g) \qquad \text{OrChopImpRule}$$

Proof:

1	$\vdash$	$f \supset f_1 \lor f_2$	given
2	$\vdash$	$f ; g \supset (f_1 \lor f_2) ; g$	1, **LeftChopImpChop**
3	$\vdash$	$(f \lor f_1) ; g \equiv f_1 ; g \lor f_2 ; g$	**OrChopEqv**
4	$\vdash$	$f ; g \supset (f_1 ; g) \lor (f_2 ; g)$	2, 3, **Prop**

qed

$$\vdash f \equiv f \lor f_1 \quad \Rightarrow \quad \vdash f ; g \equiv (f_1 ; g) \lor (f_2 ; g) \qquad \text{OrChopEqvRule}$$

Proof:

1	$\vdash$	$f \equiv f_1 \lor f_2$	given
2	$\vdash$	$f ; g \equiv (f_1 \lor f_2) ; g$	1, **LeftChopEqvChop**
3	$\vdash$	$(f \lor f_1) ; g \equiv f_1 ; g \lor f_2 ; g$	**OrChopEqv**
4	$\vdash$	$f ; g \equiv (f_1 ; g) \lor (f_2 ; g)$	2, 3, **EqvChain**

qed

$$\vdash f \supset g \quad \Rightarrow \quad \vdash \bigcirc f \supset \bigcirc g \qquad \text{NextImpNext}$$

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$\Box(f \supset g)$	1, BoxGen
3	$\vdash$	$\Box(f \supset g) \supset (\mathsf{skip} ; f) \supset (\mathsf{skip} ; g)$	BoxChopImpChop
4	$\vdash$	$(\mathsf{skip} ; f) \supset (\mathsf{skip} ; g)$	2, 3, MP
5	$\vdash$	$\bigcirc f \supset \bigcirc g$	4, def. of $\bigcirc$

qed

$\vdash \quad \bigcirc(f \supset g) \supset \bigcirc f \supset \bigcirc g$ NextImpDist

Proof:

1	$\vdash$	$\neg(f \supset g) \equiv f \wedge \neg g$	Prop
2	$\vdash$	$\mathsf{skip} ; \neg(f \supset g) \equiv \mathsf{skip} ; (f \wedge \neg g)$	1, RightChopEqvChop
3	$\vdash$	$f \supset g \vee (f \wedge \neg g)$	Prop
4	$\vdash$	$\mathsf{skip} ; f \supset (\mathsf{skip} ; g) \vee (\mathsf{skip} ; (f \wedge \neg g))$	3, ChopOrImpRule
5	$\vdash$	$\neg(\mathsf{skip} ; (f \wedge \neg g)) \supset (\mathsf{skip} ; f) \supset (\mathsf{skip} ; g)$	4, Prop
6	$\vdash$	$\neg(\mathsf{skip} ; \neg(f \supset g)) \supset (\mathsf{skip} ; f) \supset (\mathsf{skip} ; g)$	2, 5, Prop
7	$\vdash$	$\neg\bigcirc\neg(f \supset g) \supset \bigcirc f \supset \bigcirc g$	6, def. of $\bigcirc$
8	$\vdash$	$\bigcirc(f \supset g) \supset \neg\bigcirc\neg(f \supset g)$	NextImpNotNextNot
9	$\vdash$	$\bigcirc(f \supset g) \supset \bigcirc f \supset \bigcirc g$	7, 8, Prop

qed

$\vdash \quad f ; g \supset \Diamond g$ ChopImpDiamond

Proof:

1	$\vdash$	$f \supset \mathsf{true}$	Prop
2	$\vdash$	$f ; g \supset \mathsf{true} ; g$	1, LeftChopImpChop
3	$\vdash$	$f ; g \supset \Diamond g$	2, def. of $\Diamond$

qed

$\vdash \quad f \supset \Diamond f$ NowImpDiamond

Proof:

1	$\vdash$	$\mathsf{empty} ; f \equiv f$	EmptyChop
2	$\vdash$	$\mathsf{empty} \supset \mathsf{true}$	Prop
3	$\vdash$	$\mathsf{empty} ; f \supset \mathsf{true} ; f$	2, LeftChopImpChop
4	$\vdash$	$f \supset \mathsf{true} ; f$	1, 3, Prop
5	$\vdash$	$f \supset \Diamond f$	4, def. of $\Diamond$

qed

$\vdash \quad \bigcirc\Diamond f \supset \Diamond f$ NextDiamondImpDiamond

Proof:

1	$\vdash$	$(\mathsf{skip}\ ;\ \mathsf{true})\ ;\ f \equiv \mathsf{skip}\ ;\ (\mathsf{true}\ ;\ f)$	ChopAssoc
2	$\vdash$	$(\mathsf{skip}\ ;\ \mathsf{true})\ ;\ f \equiv \bigcirc\Diamond f$	1, def. of $\bigcirc, \Diamond$
3	$\vdash$	$(\mathsf{skip}\ ;\ \mathsf{true})\ ;\ f \supset \Diamond f$	ChopImpDiamond
4	$\vdash$	$\bigcirc\Diamond f \supset \Diamond f$	2, 3, Prop

qed

$\vdash \quad \Box f \supset f \wedge Ⓦ\Box f$ BoxImpNowAndWeakNext

Proof:

1	$\vdash$	$\neg f \supset \Diamond\neg f$	NowImpDiamond
2	$\vdash$	$\neg\Diamond\neg f \supset f$	1, Prop
3	$\vdash$	$\Box f \supset f$	2, def. of $\Box$
4	$\vdash$	$\bigcirc\Diamond\neg f \supset \Diamond\neg f$	NextDiamondImpDiamond
5	$\vdash$	$\neg\neg\Diamond\neg f \supset \Diamond\neg f$	Prop
6	$\vdash$	$\bigcirc\neg\neg\Diamond\neg f \supset \bigcirc\Diamond\neg f$	5, NextImpNext
7	$\vdash$	$\bigcirc\neg\neg\Diamond\neg f \supset \Diamond\neg f$	4, 6, ImpChain
8	$\vdash$	$\bigcirc\neg\Box f \supset \Diamond\neg f$	7, def. of $\Box$
9	$\vdash$	$\neg\Diamond\neg f \supset \neg\bigcirc\neg\Box f$	8, Prop
10	$\vdash$	$\Box f \supset Ⓦ\Box f$	9, def. of $\Box$, Ⓦ
11	$\vdash$	$\Box f \supset f \wedge Ⓦ\Box f$	3, 10, Prop

qed

$\vdash f \supset g \quad \Rightarrow \quad \vdash \Box f \supset \Box g$ BoxImpBoxRule

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$\neg g \supset \neg f$	1, Prop
3	$\vdash$	$\Box(\neg g \supset \neg f)$	2, BoxGen
4	$\vdash$	$\Box(\neg g \supset \neg f) \supset (\mathsf{true}\ ;\ \neg g) \supset (\mathsf{true}\ ;\ \neg f)$	BoxChopImpChop
5	$\vdash$	$\mathsf{true}\ ;\ \neg g \supset \mathsf{true}\ ;\ \neg f$	3, 4, MP
6	$\vdash$	$\Diamond\neg g \supset \Diamond\neg f$	5, def. of $\Diamond$
7	$\vdash$	$\neg\Diamond\neg f \supset \neg\Diamond\neg g$	6, Prop
8	$\vdash$	$\Box f \supset \Box g$	7, def. of $\Box$

qed

$\vdash \quad \Box(f \supset g) \supset \Box f \supset \Box g$ BoxImpDist

Proof:

1	$\vdash$	$f \supset g \supset \neg g \supset \neg f$	**Prop**
2	$\vdash$	$\Box(f \supset g) \supset \Box(\neg g \supset \neg f)$	1, **BoxImpBoxRule**
3	$\vdash$	$\Box(\neg g \supset \neg f) \supset (\mathsf{true} \,;\, \neg g) \supset (\mathsf{true} \,;\, \neg f)$	**BoxChopImpChop**
4	$\vdash$	$\Box(f \supset g) \supset (\mathsf{true} \,;\, \neg g) \supset (\mathsf{true} \,;\, \neg f)$	2, 3, **Prop**
5	$\vdash$	$\Box(f \supset g) \supset \Diamond \neg g \supset \Diamond \neg f$	4, def. of $\Diamond$
6	$\vdash$	$\Box(f \supset g) \supset \neg \Diamond \neg f \supset \neg \Diamond \neg g$	5, **Prop**
7	$\vdash$	$\Box(f \supset g) \supset \Box f \supset \Box g$	6, def. of $\Box$

qed

$\vdash \Diamond\, \mathsf{empty}$ — DiamondEmpty

Proof:

1	$\vdash$	true	**Prop**
2	$\vdash$	true ; empty $\equiv$ true	**ChopEmpty**
3	$\vdash$	true ; empty	1, 2, **Prop**
4	$\vdash$	$\Diamond$ empty	3, def. of $\Diamond$

qed

Here are some use derived rules for linear time temporal logic. We omit their proofs:

$\vdash f \equiv g \quad \Rightarrow \quad \vdash \bigcirc f \equiv \bigcirc g$ — NextEqvNext

$\vdash f \wedge g \supset h \quad \Rightarrow \quad \vdash \bigcirc f \wedge \bigcirc g \supset \bigcirc h$ — NextAndNextImpNextRule

$\vdash f \wedge g \equiv h \quad \Rightarrow \quad \vdash \bigcirc f \wedge \bigcirc g \equiv \bigcirc h$ — NextAndNextEqvNextRule

$\vdash f \equiv g \quad \Rightarrow \quad \vdash ⓦ f \equiv ⓦ g$ — WeakNextEqvWeakNext

$\vdash f \supset g \quad \Rightarrow \quad \vdash \Diamond f \supset \Diamond g$ — DiamondImpDiamond

$\vdash f \equiv g \quad \Rightarrow \quad \vdash \Diamond f \equiv \Diamond g$ — DiamondEqvDiamond

$\vdash f \equiv g \quad \Rightarrow \quad \vdash \Box f \equiv \Box g$ — BoxEqvBox

$\vdash f \wedge g \supset h \quad \Rightarrow \quad \vdash \Box f \wedge \Box g \supset \Box h$ — BoxAndBoxImpBoxRule

$\vdash f \wedge g \equiv h \quad \Rightarrow \quad \vdash \Box f \wedge \Box g \equiv \Box h$ — BoxAndBoxEqvBoxRule

$\vdash f \supset g, \quad \vdash \mathsf{more} \wedge f \supset \bigcirc f \quad \Rightarrow \quad \vdash f \supset \Box g$	BoxIntro

$\vdash (f \wedge \neg g) \supset \bigcirc f \quad \Rightarrow \quad \vdash f \supset \Diamond g$	DiamondIntro

$\vdash f \supset \bigcirc f \quad \Rightarrow \quad \vdash \neg f$	NextLoop

$\vdash f \wedge \neg g \supset \bigcirc f \wedge \neg \bigcirc g \quad \Rightarrow \quad \vdash f \supset g$	NextContra

$\vdash ⓦ\Box f \supset f \quad \Rightarrow \quad \vdash f$	WeakNextBoxInduct

$\vdash \mathsf{empty} \supset f, \quad \vdash \bigcirc f \supset f \quad \Rightarrow \quad \vdash f$	EmptyNextInducta

$\vdash \mathsf{empty} \wedge f \supset g, \vdash ⓦ(f \supset g) \wedge f \supset g \quad \Rightarrow \quad \vdash f \supset g$	EmptyNextInductb

$\vdash f \supset g \quad \Rightarrow \quad \vdash \mathsf{fin}\ f \supset \mathsf{fin}\ g$	FinImpFin

$\vdash f \equiv g \quad \Rightarrow \quad \vdash \mathsf{fin}\ f \equiv \mathsf{fin}\ g$	FinEqvFin

$\vdash f \wedge g \supset h \quad \Rightarrow \quad \vdash \mathsf{fin}\ f \wedge \mathsf{fin}\ g \supset \mathsf{fin}\ h$	FinAndFinImpFinRule

$\vdash f \wedge g \equiv h \quad \Rightarrow \quad \vdash \mathsf{fin}\ f \wedge \mathsf{fin}\ g \equiv \mathsf{fin}\ h$	FinAndFinEqvFinRule

$\vdash f \equiv g \quad \Rightarrow \quad \vdash \mathsf{halt}\ f \equiv \mathsf{halt}\ g$	HaltEqvHalt

Note that **ImpChain** can be viewed as a special case of **Prop**. If desired, a deduction theorem can also be proved.
We now give proofs of some derived inference rules and theorems:

$\vdash \quad ⊡(f \supset g) \supset ◈ f \supset ◈ g$	BiImpDiImpDi

Proof:

1	$\vdash$	$⊡(f \supset g) \supset (f\,;\mathsf{true}) \supset (g\,;\mathsf{true})$	**BiChopImpChop**
2	$\vdash$	$⊡(f \supset g) \supset ◈ f \supset ◈ g$	1, def. of ◈

qed

$$\vdash f \supset g \quad \Rightarrow \quad \vdash \Diamond_i f \supset \Diamond_i g$$ DiImpDi

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$f \,;\, \mathsf{true} \supset g \,;\, \mathsf{true}$	1, **LeftChopImpChop**
3	$\vdash$	$\Diamond_i f \supset \Diamond_i g$	2, def. of $\Diamond_i$

qed

Another corollary is the following:

$$\vdash f \supset g \quad \Rightarrow \quad \vdash \Box_i f \supset \Box_i g$$ BiImpBiRule

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$\neg g \supset \neg f$	1, **Prop**
3	$\vdash$	$\Diamond_i \neg g \supset \Diamond_i \neg f$	2, **DiImpDi**
4	$\vdash$	$\neg \Diamond_i \neg f \supset \neg \Diamond_i \neg g$	3, **Prop**
5	$\vdash$	$\Box_i f \supset \Box_i g$	4, def. of $\Box_i$

qed

$$\vdash f \equiv f_1 \quad \Rightarrow \quad \vdash f \,;\, g \equiv f_1 \,;\, g$$ LeftChopEqvChop

Proof:

1	$\vdash$	$f \equiv f_1$	given
2	$\vdash$	$f \supset f_1$	1, **Prop**
3	$\vdash$	$f \,;\, g \supset f_1 \,;\, g$	2, **LeftChopImpChop**
4	$\vdash$	$f_1 \supset f$	1, **Prop**
5	$\vdash$	$f_1 \,;\, g \supset f \,;\, g$	4, **LeftChopImpChop**
6	$\vdash$	$f \,;\, g \equiv f_1 \,;\, g$	3, 5, **Prop**

qed

Here is a corollary for the operator $\Diamond_i$:

$$\vdash f \equiv g \quad \Rightarrow \quad \vdash \Diamond_i f \equiv \Diamond_i g$$ DiEqvDi

Proof:

1	$\vdash$	$f \equiv f_1$	given
2	$\vdash$	$f \,;\, \mathsf{true} \equiv g \,;\, \mathsf{true}$	1, **LeftChopEqvChop**
3	$\vdash$	$\Diamond_i f \equiv \Diamond_i g$	2, def. of $\Diamond_i$

qed

Here is a corollary for the operator $\Box_i$:

$$\vdash f \equiv g \quad \Rightarrow \quad \vdash \boxed{\mathrm{i}}\, f \equiv \boxed{\mathrm{i}}\, g$$ BiEqvBi

Proof:

1	$\vdash$	$f \equiv g$	given
2	$\vdash$	$\neg f \equiv \neg g$	1, **Prop**
3	$\vdash$	$\Diamond\mkern-11mu\mathrm{i}\mkern4mu \neg f \equiv \Diamond\mkern-11mu\mathrm{i}\mkern4mu \neg g$	2, **DiEqvDi**
4	$\vdash$	$\neg \Diamond\mkern-11mu\mathrm{i}\mkern4mu \neg f \equiv \neg \Diamond\mkern-11mu\mathrm{i}\mkern4mu \neg g$	3, **Prop**
5	$\vdash$	$\boxed{\mathrm{i}}\, f \equiv \boxed{\mathrm{i}}\, g$	4, def. of $\boxed{\mathrm{i}}$

qed

$$\vdash f\,;g \supset g \quad \Rightarrow \quad \vdash f\,;g\,;h \supset g\,;h$$ LeftChopChopImpChopRule

Proof:

1	$\vdash$	$f\,;g \supset g$	given
2	$\vdash$	$(f\,;g)\,;h \supset g\,;h$	1, **LeftChopImpChop**
3	$\vdash$	$(f\,;g)\,;h \equiv f\,;g\,;h$	**ChopAssoc**
4	$\vdash$	$f\,;g\,;h \supset g\,;h$	2, 3, **Prop**

qed

$$\vdash \quad (f \wedge f_1)\,;g \equiv (f_1 \wedge f)\,;g$$ AndChopCommute

Proof:

1	$\vdash$	$f \wedge f_1 \equiv f_1 \wedge f$	**Prop**
2	$\vdash$	$(f \wedge f_1)\,;g \equiv (f_1 \wedge f)\,;g$	1, **LeftChopEqvChop**

qed

$$\vdash w \wedge f \supset f_1 \quad \Rightarrow \quad \vdash w \wedge (f\,;g) \supset (f_1\,;g)$$ StateAndChopImpChopRule

Proof:

1	$\vdash$	$w \wedge f \supset f_1$	given
2	$\vdash$	$(w \wedge f)\,;g \supset f_1\,;g$	1, **LeftChopImpChop**
3	$\vdash$	$(w \wedge f)\,;g \equiv w \wedge (f\,;g)$	**StateAndChop**
4	$\vdash$	$w \wedge f\,;g \supset f_1\,;g$	2, 3, **Prop**

qed

$$\vdash w \supset (f \equiv f_1) \quad \Rightarrow \quad \vdash w \supset ((f\,;g) \equiv (f_1\,;g))$$ StateImpChopEqvChop

Proof:

1	$\vdash$	$w \supset f \equiv f_1$	given
2	$\vdash$	$w \wedge f \supset f_1$	1, **Prop**
3	$\vdash$	$w \wedge (f\,;g) \supset (f_1\,;g)$	2, **StateAndChopImpChopRule**
4	$\vdash$	$w \wedge f_1 \supset f$	1, **Prop**
5	$\vdash$	$w \wedge (f_1\,;g) \supset (f\,;g)$	4, **StateAndChopImpChopRule**
6	$\vdash$	$w \supset (f\,;g) \equiv (f_1\,;g)$	3, 5, **Prop**

qed

$\vdash f \equiv w \wedge f_1 \quad \Rightarrow \quad \vdash (f\,;g) \equiv w \wedge (f_1\,;g)$ ChopEqvStateAndChop

Proof:

1	$\vdash$	$f \equiv w \wedge f_1$	given
2	$\vdash$	$f\,;g \equiv (w \wedge f_1)\,;g$	1, **LeftChopEqvChop**
3	$\vdash$	$(w \wedge f_1)\,;g \equiv w \wedge (f_1\,;g)$	**StateAndChop**
4	$\vdash$	$(f\,;g) \equiv w \wedge (f_1\,;g)$	2, 3, **EqvChain**

qed

$\vdash \quad f \supset \Diamond\!\!\!\!{\scriptstyle i}\;\, f$ DiIntro

Proof:

1	$\vdash$	$f\,;\text{empty} \equiv f$	**ChopEmpty**
2	$\vdash$	$\text{empty} \supset \text{true}$	**PTL**
3	$\vdash$	$\Box(\text{empty} \supset \text{true})$	2, **BoxGen**
4	$\vdash$	$\Box(\text{empty} \supset \text{true}) \supset (f\,;\text{empty} \supset f\,;\text{true})$	**BoxChopImpChop**
5	$\vdash$	$f\,;\text{empty} \supset f\,;\text{true}$	3, 4, **MP**
6	$\vdash$	$f\,;\text{empty} \supset \Diamond\!\!\!\!{\scriptstyle i}\;\, f$	5, def. of $\Diamond\!\!\!\!{\scriptstyle i}\;\,$
7	$\vdash$	$f \supset \Diamond\!\!\!\!{\scriptstyle i}\;\, f$	1, 6, **Prop**

qed

The following is a corollary of **DiIntro**:

$\vdash \quad \boxed{i}\, f \supset f$ BiElim

Proof:

1	$\vdash$	$\neg f \supset \Diamond\!\!\!\!{\scriptstyle i}\;\, \neg f$	**DiIntro**
2	$\vdash$	$(\neg f \supset \Diamond\!\!\!\!{\scriptstyle i}\;\, \neg f) \supset (\neg \Diamond\!\!\!\!{\scriptstyle i}\;\, \neg f \supset f)$	**Prop**
3	$\vdash$	$\neg \Diamond\!\!\!\!{\scriptstyle i}\;\, \neg f \supset f$	1, 2, **MP**
4	$\vdash$	$\boxed{i}\, f \supset f$	3, def. of $\boxed{i}$

qed

The following is used in the proof of lemma **BiImpDist**:

$\vdash \quad \boxed{i}(\neg g \supset \neg f) \supset (\boxed{i}\, f) \supset (\boxed{i}\, g)$ BiContraPosImpDist

Proof:

1	$\vdash$	$\boxdot(\neg g \supset \neg f) \supset (\diamondsuit \neg g) \supset (\diamondsuit \neg f)$	**BiImpDiImpDi**
2	$\vdash$	$\boxdot(\neg g \supset \neg f) \supset (\neg \diamondsuit \neg f) \supset (\neg \diamondsuit \neg g)$	1, **Prop**
3	$\vdash$	$\boxdot(\neg g \supset \neg f) \supset (\boxdot f) \supset (\boxdot g)$	2, def. of $\boxdot$

qed

> $\vdash \quad \boxdot(f \supset g) \supset (\boxdot f) \supset (\boxdot g)$ BiImpDist

Proof:

1	$\vdash$	$(f \supset g) \supset (\neg g \supset \neg f)$	**Prop**
2	$\vdash$	$\neg(\neg g \supset \neg f) \supset \neg(f \supset g)$	1, **Prop**
3	$\vdash$	$\boxdot(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$	2, **BiGen**
4	$\vdash$	$\boxdot(\neg(\neg g \supset \neg f) \supset \neg(f \supset g)) \supset$	
		$\boxdot(f \supset g) \supset \boxdot(\neg g \supset \neg f)$	**BiContraPosImpDist**
5	$\vdash$	$\boxdot(f \supset g) \supset \boxdot(\neg g \supset \neg f)$	3, 4, **MP**
6	$\vdash$	$\boxdot(\neg g \supset \neg f) \supset (\boxdot f) \supset (\boxdot g)$	**BiContraPosImpDist**
7	$\vdash$	$\boxdot(f \supset g) \supset (\boxdot f) \supset (\boxdot g)$	5, 6, **ImpChain**

qed

> $\vdash \quad (f \vee f_1) ; g \equiv f ; g \vee f_1 ; g$ OrChopEqv

The proof for $\subset$ is immediate from axiom **OrChopImp**.

Here is the proof for the converse:

1	$\vdash$	$f \supset f \vee f_1$	**Prop**
2	$\vdash$	$f ; g \supset (f \vee f_1) ; g$	1, **LeftChopImpChop**
3	$\vdash$	$f_1 \supset f \vee f_1$	**Prop**
4	$\vdash$	$f_1 ; g \supset (f \vee f_1) ; g$	3, **LeftChopImpChop**
5	$\vdash$	$f ; g \vee f ; g_1 \supset (f \vee f_1) ; g$	2, 4, **Prop**

qed

> $\vdash f \equiv \text{if } w \text{ then } f_1 \text{ else } f_2 \quad \Rightarrow \quad \vdash f ; g \equiv \text{if } w \text{ then } (f_1 ; g) \text{ else } (f_2 ; g)$ IfChopEqvRule

Proof:

1	$\vdash$	$f \equiv \text{if } w \text{ then } f_1 \text{ else } f_2$	given
2	$\vdash$	$f \equiv (w \wedge f_1) \vee (\neg w \wedge f_2)$	1, **Prop**
3	$\vdash$	$f ; g \equiv (w \wedge f_1) ; g \vee (\neg w \wedge f_2) ; g$	2, **OrChopEqvRule**
4	$\vdash$	$(w \wedge f_1) ; g \equiv w \wedge (f_1 ; g)$	**StateAndChop**
5	$\vdash$	$(\neg w \wedge f_2) ; g \equiv \neg w \wedge (f_2 ; g)$	**StateAndChop**
6	$\vdash$	$f ; g \equiv (w \wedge f_1 ; g) \vee (\neg w \wedge f_2 ; g)$	3, 4, 5, **Prop**
7	$\vdash$	$f ; g \equiv \text{if } w \text{ then } f_1 ; g \text{ else } f_2 ; g$	6, **Prop**

qed

$\vdash g \supset g_1 \vee g_2 \quad \Rightarrow \quad \vdash f\,;g \supset (f\,;g_1) \vee (f\,;g_2)$	ChopOrImpRule

Proof:

1	$\vdash$	$g \supset g_1 \vee g_2$	given
2	$\vdash$	$f\,;g \supset f\,;(g_1 \vee g_2)$	1, **RightChopImpChop**
3	$\vdash$	$f\,;(g \vee g) \equiv f\,;g_1 \vee f\,;g_2$	**ChopOrEqv**
4	$\vdash$	$f\,;g \supset (f\,;g_1) \vee (f\,;g_2)$	2, 3, **Prop**

qed

$\vdash g \equiv g_1 \vee g_2 \quad \Rightarrow \quad \vdash f\,;g \equiv (f\,;g_1) \vee (f\,;g_2)$	ChopOrEqvRule

Proof:

1	$\vdash$	$g \equiv g_1 \vee g_2$	given
2	$\vdash$	$f\,;g \equiv f\,;(g_1 \vee g_2)$	1, **RightChopEqvChop**
3	$\vdash$	$(f \vee f_1)\,;g \equiv f_1\,;g \vee f_2\,;g$	**ChopOrEqv**
4	$\vdash$	$f\,;g \equiv (f_1\,;g) \vee (f_2\,;g)$	2, 3, **EqvChain**

qed

$\vdash \quad (\mathrm{empty} \vee f)\,;g \equiv g \vee (f\,;g)$	EmptyOrChopEqv

Proof:

1	$\vdash$	$(\mathrm{empty} \vee f)\,;g \equiv (\mathrm{empty}\,;g) \vee (f\,;g)$	**OrChopEqv**
2	$\vdash$	$\mathrm{empty}\,;g \equiv g$	**EmptyChop**
3	$\vdash$	$(\mathrm{empty} \vee f)\,;g \equiv g \vee (f\,;g)$	1, 2, **Prop**

qed

$\vdash \quad (\mathrm{empty} \vee \bigcirc f)\,;g \equiv g \vee \bigcirc(f\,;g)$	EmptyOrNextChopEqv

Proof:

1	$\vdash$	$(\mathrm{empty} \vee \bigcirc f)\,;g \equiv g \vee ((\bigcirc f)\,;g)$	**EmptyOrChopEqv**
2	$\vdash$	$(\bigcirc f)\,;g \equiv \bigcirc(f\,;g)$	**NextChop**
3	$\vdash$	$(\mathrm{empty} \vee \bigcirc f)\,;g \equiv g \vee \bigcirc(f\,;g)$	1, 2, **Prop**

qed

$\vdash f \supset \mathrm{empty} \vee f_1 \quad \Rightarrow \quad \vdash f\,;g \supset g \vee (f_1\,;g)$	EmptyOrChopImpRule

Proof:

1	$\vdash$	$f \supset \mathrm{empty} \vee f_1$	given
2	$\vdash$	$f\,;g \supset (\mathrm{empty} \vee f_1)\,;g$	1, **LeftChopImpChop**
3	$\vdash$	$(\mathrm{empty} \vee f_1)\,;g \equiv g \vee (f_1\,;g)$	**EmptyOrChopEqv**
4	$\vdash$	$f\,;g \supset g \vee (f_1\,;g)$	2, 3, **Prop**

qed

Here is a related lemma:

$$\vdash f \equiv \mathsf{empty} \lor f_1 \quad \Rightarrow \quad \vdash f\,;g \equiv g \lor (f_1\,;g) \qquad \text{EmptyOrChopEqvRule}$$

Proof:

1	$\vdash$	$f \equiv \mathsf{empty} \lor f_1$	given
2	$\vdash$	$f\,;g \equiv (\mathsf{empty} \lor f)\,;g$	1, **LeftChopEqvChop**
3	$\vdash$	$(\mathsf{empty} \lor f)\,;g \equiv g \lor (f\,;g)$	**EmptyOrChopEqv**
4	$\vdash$	$f\,;g \equiv g \lor (f\,;g)$	2, 3, **Prop**

qed

The following is a useful special case of **EmptyOrChopImpRule**:

$$\vdash f \supset \mathsf{empty} \lor \bigcirc f_1 \quad \Rightarrow \quad \vdash f\,;g \supset g \lor \bigcirc(f_1\,;g) \qquad \text{EmptyOrNextChopImpRule}$$

Proof:

1	$\vdash$	$f \supset \mathsf{empty} \lor \bigcirc f_1$	given
2	$\vdash$	$f\,;g \supset (\mathsf{empty} \lor \bigcirc f_1)\,;g$	1, **LeftChopImpChop**
3	$\vdash$	$(\mathsf{empty} \lor \bigcirc f_1)\,;g \equiv g \lor \bigcirc(f_1\,;g)$	**EmptyOrNextChopEqv**
4	$\vdash$	$f\,;g \supset g \lor \bigcirc(f_1\,;g)$	2, 3, **Prop**

qed

The following an analogous special case of **EmptyOrChopEqvRule**:

$$\vdash f \equiv \mathsf{empty} \lor \bigcirc f_1 \quad \Rightarrow \quad \vdash f\,;g \equiv g \lor \bigcirc(f_1\,;g) \qquad \text{EmptyOrNextChopEqvRule}$$

Proof:

1	$\vdash$	$f \equiv \mathsf{empty} \lor \bigcirc f_1$	given
2	$\vdash$	$f\,;g \equiv (\mathsf{empty} \lor \bigcirc f_1)\,;g$	1, **LeftChopEqvChop**
3	$\vdash$	$(\mathsf{empty} \lor \bigcirc f_1)\,;g \equiv g \lor \bigcirc(f_1\,;g)$	**EmptyOrNextChopEqv**
4	$\vdash$	$f\,;g \equiv g \lor \bigcirc(f_1\,;g)$	2, 3, **Prop**

qed

Here is a corollary of **ChopOrImpRule**:

$$\vdash g \supset \mathsf{empty} \lor g_1 \quad \Rightarrow \quad \vdash f\,;g \supset f \lor (f\,;g_1) \qquad \text{ChopEmptyOrImpRule}$$

Proof:

1	$\vdash$	$g \supset \mathsf{empty} \lor g_1$	given
2	$\vdash$	$f\,;g \supset (f\,;\mathsf{empty}) \lor (f\,;g_1)$	1, **ChopOrImpRule**
3	$\vdash$	$f\,;\mathsf{empty} \equiv f$	**ChopEmpty**
4	$\vdash$	$f\,;g \supset f \lor (f\,;g_1)$	2, 3, **Prop**

qed

$\vdash \quad \Box\, w \,;\, \Box\, w \equiv \Box\, w$ BoxStateChopBoxEqvBox

Proof for $\supset$:

1	$\vdash$	$\Box\, w \equiv w \wedge (\text{empty} \vee \bigcirc\Box\, w)$	**PTL**
2	$\vdash$	$\Box\, w \,;\, \Box\, w \equiv w \wedge ((\text{empty} \vee \bigcirc\Box\, w) \,;\, \Box\, w)$	1, **ChopEqvStateAndChop**
3	$\vdash$	$(\text{empty} \vee \bigcirc\Box\, w) \,;\, \Box\, w \equiv \Box\, w \vee \bigcirc(\Box\, w \,;\, \Box\, w)$	**EmptyOrNextChopEqv**
4	$\vdash$	$\Box\, w \,;\, \Box\, w \equiv w \wedge (\Box\, w \vee \bigcirc(\Box\, w \,;\, \Box\, w))$	2, 3, **Prop**
5	$\vdash$	$\neg\Box\, w \supset \neg w \vee \neg\bigcirc\Box\, w$	**PTL**
6	$\vdash$	$(\Box\, w \,;\, \Box\, w) \wedge \neg\Box\, w \supset \bigcirc(\Box\, w \,;\, \Box\, w) \wedge \neg\bigcirc\Box$	4, 5, **Prop**
7	$\vdash$	$\Box\, w \,;\, \Box\, w \supset \Box\, w$	6, **NextContra**

qed

Proof for $\subset$:

1	$\vdash$	$\Box\, w \equiv w \wedge \Box\, w$	**PTL**
2		$\text{empty} \,;\, \Box\, w \equiv \Box\, w$	**EmptyChop**
3	$\vdash$	$(w \wedge \text{empty}) \,;\, \Box\, w \equiv w \wedge (\text{empty} \,;\, \Box\, w)$	**StateAndChop**
4	$\vdash$	$\Box\, w \equiv (w \wedge \text{empty}) \,;\, \Box\, w$	1, 2, 3, **Prop**
5	$\vdash$	$w \wedge \text{empty} \supset \Box\, w$	**PTL**
6	$\vdash$	$(w \wedge \text{empty}) \,;\, \Box\, w \supset \Box\, w \,;\, \Box\, w$	5, **LeftChopImpChop**
7	$\vdash$	$\Box\, w \supset \Box\, w \,;\, \Box\, w$	4, 6, **Prop**

qed

$\vdash \quad \neg\Box\, w \supset (\Box\, w) \rightsquigarrow \neg\Box\, w$ NotBoxStateImpBoxYieldsNotBox

Proof:

1	$\vdash$	$\Box\, w \,;\, \Box\, w \equiv \Box\, w$	**BoxStateChopBoxEqvBox**
2	$\vdash$	$\Box\, w \equiv \neg\neg\Box\, w$	**Prop**
3	$\vdash$	$\Box\, w \,;\, \Box\, w \equiv \Box\, w \,;\, \neg\neg\Box\, w$	2, **RightChopEqvChop**
4	$\vdash$	$\neg\Box\, w \supset \neg(\Box\, w \,;\, \neg\neg\Box\, w)$	1, 3, **Prop**
5	$\vdash$	$\neg\Box\, w \supset (\Box\, w) \rightsquigarrow \neg\Box\, w$	4, def. of $\rightsquigarrow$

qed

$\vdash \quad \Box\, w \wedge (f \,;\, g) \equiv (\Box\, w \wedge f) \,;\, (\Box\, w \wedge g)$ BoxStateAndChopEqvChop

Proof for $\supset$:

1	$\vdash$	$\Box\, w \equiv \boxed{a}\,\Box\, w$	**BoxStateEqvBaBoxState**
2	$\vdash$	$\Box\, w \wedge (f \,;\, g) \supset (\Box\, w \wedge f) \,;\, (\Box\, w \wedge g)$	1, **BaAndChopImport**

qed

Proof for $\subset$:

1	$\vdash$	$(\Box w \wedge f) ; (\Box w \wedge g) \supset (\Box w) ; (\Box w \wedge g)$	AndChopA
2	$\vdash$	$(\Box w) ; (\Box w \wedge g) \supset (\Box w) ; (\Box w)$	ChopAndA
3	$\vdash$	$(\Box w) ; (\Box w) \equiv \Box w$	BoxStateChopBoxEqvBox
4	$\vdash$	$(\Box w \wedge f) ; (\Box w \wedge g) \supset f ; (\Box w \wedge g)$	AndChopB
5	$\vdash$	$f ; (\Box w \wedge g) \supset f ; g$	ChopAndB
6	$\vdash$	$(\Box w \wedge f) ; (\Box w \wedge g) \supset \Box w \wedge (f ; g)$	1, 2, 3, 4, 5, Prop

qed

See also the lemma BoxStateAndCSEqvCS for *chop-star*.

$\vdash \quad w \equiv \boxdot\, w$ StateEqvBi

Proof:

1	$\vdash$	$w \supset \boxdot\, w$	StateImpBi
2	$\vdash$	$\boxdot\, w \supset w$	BiElim
3	$\vdash$	$w \equiv \boxdot\, w$	1, 2, Prop

qed

$\vdash \quad \Diamonddot\, \neg f \equiv \neg \boxdot\, f$ DiNotEqvNotBi

Proof:

1	$\vdash$	$\boxdot\, f \equiv \neg \Diamonddot\, \neg f$	def. of $\boxdot$
2	$\vdash$	$\Diamonddot\, \neg f \equiv \neg \boxdot\, f$	1, Prop

qed

$\vdash \quad \Diamonddot\, f \equiv \neg \boxdot\, \neg f$ DiEqvNotBiNot

Proof:

1	$\vdash$	$\boxdot\, \neg f \equiv \neg \Diamonddot\, \neg\neg f$	def. of $\boxdot$
2	$\vdash$	$\Diamonddot\, \neg\neg f \equiv \neg \boxdot\, \neg f$	1, Prop
3		$f \equiv \neg\neg f$	Prop
4	$\vdash$	$\Diamonddot\, f \equiv \Diamonddot\, \neg\neg f$	3, DiEqvDi
5	$\vdash$	$\Diamonddot\, f \equiv \neg \boxdot\, \neg f$	2, 4, EqvChain

qed

$\vdash \quad \Box h \wedge f ; g \supset f ; (h \wedge g)$ BoxAndChopImport

Proof:

1	$\vdash$	$h \supset g \;\supset\; (h \wedge g)$	Prop
2	$\vdash$	$\Box h \supset \Box(g \;\supset\; (h \wedge g))$	1, BoxImpBoxRule
3	$\vdash$	$\Box(g \;\supset\; (h \wedge g)) \supset f ; g \;\supset\; f ; (h \wedge g)$	BoxChopImpChop
4	$\vdash$	$\Box h \wedge f ; g \;\supset\; f ; (h \wedge g)$	2, 3, Prop

qed

$\vdash \quad f\,;g \wedge \Box\, h \supset f\,;(g \wedge h)$ ChopAndBoxImport

Proof:

1	$\vdash$	$\Box\, h \wedge f\,;g \supset f\,;(h \wedge g)$	**BoxAndChopImport**
2	$\vdash$	$f\,;(h \wedge g) \equiv f\,;(g \wedge h)$	**ChopAndCommute**
3	$\vdash$	$\Box\, h \wedge f\,;g \supset f\,;(g \wedge h)$	1, 2, **Prop**

qed

The following are easily proved:

$\vdash \quad f\,;(g \wedge g_1) \supset f\,;g$ ChopAndA

$\vdash \quad f\,;(g \wedge g_1) \supset f\,;g_1$ ChopAndB

$\vdash \quad f\,;(g \wedge g_1) \equiv f\,;(g_1 \wedge g)$ ChopAndCommute

$\vdash \quad (f \wedge g)\,;(f_1 \wedge g_1) \equiv (g \wedge f)\,;(g_1 \wedge f_1)$ AndChopAndCommute

Proof:

1	$\vdash$	$(f \wedge g)\,;(f_1 \wedge g_1) \equiv (g \wedge f)\,;(f_1 \wedge g_1)$	**AndChopCommute**
2	$\vdash$	$(g \wedge f)\,;(f_1 \wedge g_1) \equiv (g \wedge f)\,;(g_1 \wedge f_1)$	**ChopAndCommute**
3	$\vdash$	$(f \wedge g)\,;(f_1 \wedge g_1) \equiv (g \wedge f)\,;(g_1 \wedge f_1)$	1, 2, **EqvChain**

qed

$\vdash f \supset f_1, \quad \vdash g \supset g_1 \quad \Rightarrow \quad \vdash f\,;g \supset f_1\,;g_1$ ChopImpChop

Proof:

1	$\vdash$	$f \supset f_1$	given
2	$\vdash$	$f\,;g \supset f_1\,;g$	1, **LeftChopImpChop**
3	$\vdash$	$g \supset g_1$	given
4	$\vdash$	$f_1\,;g \supset f_1\,;g_1$	3, **RightChopImpChop**
5	$\vdash$	$f\,;g \supset f_1\,;g_1$	2, 4, **ImpChain**

qed

$\vdash f \equiv f_1, \quad \vdash g \equiv g_1 \quad \Rightarrow \quad \vdash f\,;g \equiv f_1\,;g_1$ ChopEqvChop

Proof:

1	$\vdash$	$f \equiv f_1$	given
2	$\vdash$	$f\,;g \equiv f_1\,;g$	1, **LeftChopEqvChop**
3	$\vdash$	$g \equiv g_1$	given
4	$\vdash$	$f_1\,;g \equiv f_1\,;g_1$	3, **RightChopEqvChop**
5	$\vdash$	$f\,;g \equiv f_1\,;g_1$	2, 4, **EqvChain**

qed

$$\vdash f_1 \supset g_1, \ldots, \vdash f_n \supset g_n \quad \Rightarrow \quad \vdash (f_1\,;\ldots;f_n) \supset (g_1\,;\ldots g_n) \qquad \text{MultChopImpChop}$$

Proof is by induction on n.

Proof for $n = 1$:

1	$f_1 \supset g_1$	given

qed

Proof for $n > 1$:

1	$\vdash$	$f_1 \supset g_1$	given
2	$\vdash$	$f_i \supset g_i$, for $2 \le i \le n$	given
3	$\vdash$	$(f_2\,;\ldots;f_n) \supset (g_2\,;\ldots g_n)$	2, *induction*
4	$\vdash$	$f_1\,;f_2\,;\ldots;f_n \supset g_1\,;g_2\,;\ldots g_n$	1, 3, **ChopImpChop**

qed

$$\vdash \quad \Box\, h \supset f\,;g \;\supset\; f\,;(\Box\, h \wedge g) \qquad \text{BoxChopImpChopBox}$$

Proof:

1	$\vdash$	$\Box\, h \supset \Box(g \;\supset\; \Box\, h \wedge g)$	**PTL**
2	$\vdash$	$\Box(g \;\supset\; \Box\, h \wedge g) \supset f\,;g \;\supset\; f\,;(\Box\, h \wedge g)$	**BoxChopImpChop**
3	$\vdash$	$\Box\, h \supset f\,;g \;\supset\; f\,;(\Box\, h \wedge g)$	1, 2, **Prop**

qed

$$\vdash \quad \neg(f\,;g) \equiv f \mathrel{\rightsquigarrow} \neg g \qquad \text{NotChopEqvYieldsNot}$$

Proof:

1	$\vdash$	$g \equiv \neg\neg g$	**Prop**
2	$\vdash$	$f\,;g \equiv f\,;\neg\neg g$	1, **RightChopEqvChop**
3	$\vdash$	$\neg(f\,;g) \equiv \neg(f\,;\neg\neg g)$	2, **Prop**
4	$\vdash$	$\neg(f\,;g) \equiv \neg(f \mathrel{\rightsquigarrow} g)$	def. of $\rightsquigarrow$

qed

The following lemma **TrueChopEqvDiamond** is no longer needed since $\Diamond$ is now defined in terms of *chop:*

$\vdash \quad \mathsf{true}\,;f \equiv \Diamond f$	TrueChopEqvDiamond

$\vdash \quad \Diamond f \supset \mathsf{true}\,;f$	DiamondImpTrueChop

Proof:

1	$\vdash$	$\Diamond f \supset f \lor \bigcirc\Diamond f$	**PTL**
2	$\vdash$	$\mathsf{true} \equiv \mathsf{empty} \lor \bigcirc\mathsf{true}$	**PTL**
3	$\vdash$	$\mathsf{true}\,;f \equiv (\mathsf{empty} \lor \bigcirc\mathsf{true})\,;f$	2, **LeftChopEqvChop**
4	$\vdash$	$(\mathsf{empty} \lor \bigcirc\mathsf{true})\,;f \equiv \mathsf{empty}\,;f \lor (\bigcirc\mathsf{true})\,;f$	**OrChopEqv**
5	$\vdash$	$\mathsf{empty}\,;f \equiv f$	**EmptyChop**
6	$\vdash$	$(\bigcirc\mathsf{true})\,;f \equiv \bigcirc(\mathsf{true}\,;f)$	**NextChop**
7	$\vdash$	$\mathsf{true}\,;f \equiv f \lor \bigcirc(\mathsf{true}\,;f)$	3, 4, 5, 6, **Prop**
8	$\vdash$	$\Diamond f \land \neg(\mathsf{true}\,;f) \supset \bigcirc\Diamond f \land \neg\bigcirc(\mathsf{true}\,;f)$	1, 7, **Prop**
9	$\vdash$	$\Diamond f \supset \mathsf{true}\,;f$	8, **NextContra**

qed

$\vdash \quad \boxed{i}\, f \land (f_1\,;g) \supset (f \land f_1)\,;g$	BiAndChopImport

Proof:

1	$\vdash$	$f \supset (f_1 \ \supset\ f \land f_1)$	**Prop**
2	$\vdash$	$\boxed{i}\, f \supset \boxed{i}(f_1 \ \supset\ f \land f_1)$	1, **BiImpBiRule**
3	$\vdash$	$\boxed{i}(f_1 \ \supset\ f \land f_1) \supset f\,;g \ \supset\ (f \land f_1)\,;g$	**BiChopImpChop**
4	$\vdash$	$f\,;g \supset (f \land f_1)\,;g$	1, 3, **MP**

qed

2.2 Further properties of Diamond-i and Box-i

$\vdash \quad f \supset \Diamond_i\, f$	ImpDi

Proof:

1	$\vdash$	$f\,;\mathsf{empty} \equiv f$	**ChopEmpty**
2	$\vdash$	$\mathsf{empty} \supset \mathsf{true}$	**Prop**
3	$\vdash$	$f\,;\mathsf{empty} \supset f\,;\mathsf{true}$	2, **RightChopImpChop**
4	$\vdash$	$f \supset f\,;\mathsf{true}$	1, 3, **Prop**
5	$\vdash$	$f \supset \Diamond_i\, f$	4, def. of $\Diamond_i$

qed

$\vdash \quad \neg\Diamond_i\, \mathsf{false}$	NotDiFalse

Proof:

1	$\vdash$	$\text{true} \supset \Box_i\,\text{true}$	**StateImpBi**
2	$\vdash$	true	**Prop**
3	$\vdash$	$\Box_i\,\text{true}$	1, 2, **MP**
4	$\vdash$	$\neg \Diamond_i \neg\text{true}$	3, def. of $\Box_i$
5	$\vdash$	$\neg\text{true} \equiv \text{false}$	**Prop**
6	$\vdash$	$\Diamond_i \neg\text{true} \equiv \Diamond_i\,\text{false}$	5, **DiEqvDi**
7	$\vdash$	$\neg \Diamond_i\,\text{false}$	4, 6, **Prop**

qed

> $\vdash \quad \Diamond_i\, w \equiv w$ — DiState

Proof for $\supset$:

1	$\vdash$	$\neg w \supset \Box_i \neg w$	**StateImpBi**
2	$\vdash$	$\neg w \supset \neg \Diamond_i \neg\neg w$	1, def. of $\Box_i$
3	$\vdash$	$(\neg w \supset \neg \Diamond_i \neg\neg w) \supset (\Diamond_i \neg\neg w \supset w)$	**Prop**
4	$\vdash$	$\Diamond_i \neg\neg w \supset w$	2, 3, **MP**
5	$\vdash$	$w \supset \neg\neg w$	**Prop**
6	$\vdash$	$\Diamond_i\, w \supset \Diamond_i \neg\neg w$	**DiImpDi**
7	$\vdash$	$\Diamond_i\, w \supset w$	4, 6, **ImpChain**

qed

Proof for $\subset$:

1	$w \supset \Diamond_i\, w$	**ImpDi**

qed

Here are two important corollaries of **DiState** that are easy to prove:

> $\vdash \quad w ; f \supset w.$ — StateChop

> $\vdash \quad (w \wedge f) ; g \supset w$ — StateChopExportA

The following lets us move a state formula into the left side of a *chop*:

> $\vdash \quad w \wedge (f ; g) \supset (w \wedge f) ; g$ — StateAndChopImport

Proof:

1	$\vdash$	$w \supset \Box_i\, w$	**StateImpBi**
2	$\vdash$	$w \wedge (f ; g) \supset \Box_i\, w \wedge (f ; g)$	1, **Prop**
3	$\vdash$	$\Box_i\, w \wedge (f ; g) \supset (w \wedge f) ; g$	**BiAndChopImport**
4	$\vdash$	$w \wedge (f ; g) \supset (w \wedge f) ; g$	2, 3, **ImpChain**

qed

With this proved, we can easily combine it with theorem **StateChopExportA** to deduce the following equivalence:

$\vdash \quad (w \wedge f) ; g \equiv w \wedge (f ; g)$ StateAndChop

A useful corollary used in decomposing the left side of *chop*:

$\vdash \quad (w \wedge \mathsf{empty}) ; f \equiv w \wedge f$ StateAndEmptyChop

Proof:

1	$\vdash$	$(w \wedge \mathsf{empty}) ; f \equiv w \wedge \mathsf{empty} ; f$	**StateAndChop**
2	$\vdash$	$\mathsf{empty} ; f \equiv f$	**EmptyChop**
3	$\vdash$	$(w \wedge \mathsf{empty}) ; f \equiv w \wedge f$	1, 2, **Prop**

qed

$\vdash \quad (w \wedge \bigcirc f) ; g \equiv w \wedge \bigcirc(f ; g)$ StateAndNextChop

Proof:

1	$\vdash$	$(w \wedge \bigcirc f) ; g \equiv w \wedge (\bigcirc f) ; g$	**StateAndChop**
2	$\vdash$	$(\bigcirc f) ; g \equiv \bigcirc(f ; g)$	**NextChop**
3	$\vdash$	$(w \wedge \bigcirc f) ; g \equiv w \wedge \bigcirc(f ; g)$	1, 2, **Prop**

qed

$\vdash \quad \bigcirc((w \wedge f) ; g) \equiv \bigcirc w \wedge \bigcirc(f ; g)$ NextStateAndChop

Proof:

1	$\vdash$	$(w \wedge f) ; g \equiv w \wedge f ; g$	**StateAndChop**
2	$\vdash$	$\bigcirc((w \wedge f) ; g) \equiv \bigcirc(w \wedge f ; g)$	1, **NextEqvNext**
3	$\vdash$	$\bigcirc(w \wedge f ; g) \equiv \bigcirc w \wedge \bigcirc(f ; g)$	**PTL**
4	$\vdash$	$\bigcirc((w \wedge f) ; g) \equiv \bigcirc w \wedge \bigcirc(f ; g)$	2, 3, **EqvChain**

qed

$\vdash \quad (w \supset (f \mathrel{\rightsquigarrow\!\!\gg} g)) \equiv (w \wedge f) \mathrel{\rightsquigarrow\!\!\gg} g$ StateYieldsEqv

Proof:

1	$\vdash$	$w \wedge f ; (\neg g) \equiv (w \wedge f) ; \neg g$	**StateAndChop**
2	$\vdash$	$(w \supset \neg(f ; \neg g)) \equiv \neg((w \wedge f) ; \neg g)$	1, **Prop**

qed

$$\vdash \quad w \wedge \Diamond f \equiv \Diamond(w \wedge f)$$ StateAndDi

Proof:

1	$\vdash$	$w \wedge f \mathbin{;} \mathrm{true} \equiv (w \wedge f) \mathbin{;} \mathrm{true}$	**StateAndChop**
2	$\vdash$	$w \wedge \Diamond f \equiv \Diamond(w \wedge f)$	1, def. of $\Diamond$

qed

$$\vdash \quad \Diamond \bigcirc f \equiv \bigcirc \Diamond f$$ DiNext

Proof:

1	$\vdash$	$(\bigcirc f) \mathbin{;} \mathrm{true} \equiv \bigcirc(f \mathbin{;} \mathrm{true})$	**NextChop**
2	$\vdash$	$\Diamond \bigcirc f \equiv \bigcirc \Diamond f$	1, def. of $\Diamond$

qed

$$\vdash \quad \Diamond \bigcirc w \equiv \bigcirc w$$ DiNextState

Proof of $\supset$:

1	$\vdash$	$\Diamond \bigcirc w \equiv \bigcirc \Diamond w$	**DiNext**
2	$\vdash$	$\Diamond w \equiv w$	**DiNextState**
3	$\vdash$	$\bigcirc \Diamond w \equiv \bigcirc w$	**NextEqvNext**
4	$\vdash$	$\Diamond \bigcirc w \equiv \bigcirc w$	1, 3, 4, **EqvChain**

qed

$$\vdash w \supset f \quad \Rightarrow \quad \vdash w \supset \boxdot f$$ StateImpBiGen

Proof:

1	$\vdash$	$w \supset f$	given
2	$\vdash$	$\neg f \supset \neg w$	1, **Prop**
3	$\vdash$	$\Diamond \neg f \supset \Diamond \neg w$	2, **DiImpDi**
4	$\vdash$	$\Diamond \neg w \equiv \neg w$	**DiState**
5	$\vdash$	$\Diamond \neg f \supset \neg w$	3, 4, **Prop**
6	$\vdash$	$w \supset \neg \Diamond \neg f$	5, **Prop**
7	$\vdash$	$w \supset \boxdot f$	6, def. of $\boxdot$

qed

Let us now consider the following valuable theorem:

$$\vdash \quad f \mathbin{;} g \wedge \neg(f \mathbin{;} g_1) \supset f \mathbin{;} (g \wedge \neg g_1)$$ ChopAndNotChopImp

Proof:

1	$\vdash$	$g \supset (g \wedge \neg g_1) \vee g_1$	**Prop**
2	$\vdash$	$f ; g \supset f ; (g \wedge \neg g_1) \vee f ; g_1$	1, **LeftChopImpChop**
3	$\vdash$	$f ; g \wedge \neg(f ; g_1) \supset f ; (g \wedge \neg g_1)$	2, **Prop**

qed

Here is a related theorem for the *yields* operator:

$$\vdash \quad f ; g \wedge f \leadsto g_1 \supset f ; (g \wedge g_1)$$ ChopAndYieldsImp

This shows how *yields* adds information to the right of a suitable *chop* formula.

Proof:

1	$\vdash$	$g \supset (g \wedge g_1) \vee \neg g_1$	**Prop**
2	$\vdash$	$f ; g \supset f ; (g \wedge g_1) \vee f ; \neg g_1$	1, **LeftChopImpChop**
3	$\vdash$	$f ; g \wedge \neg(f ; \neg g_1) \supset f ; (g \wedge g_1)$	2, **Prop**
4	$\vdash$	$f ; g \wedge f \leadsto g_1 \supset f ; (g \wedge g_1)$	3, def. of $\leadsto$

qed

Here is a corollary:

$$\vdash \quad f ; g \wedge f \leadsto (g \supset g_1) \supset f ; g_1$$ ChopAndYieldsMP

Proof:

1	$\vdash$	$f ; g \wedge f \leadsto (g \supset g_1) \supset f ; (g \wedge (g \supset g_1))$	**ChopAndYieldsImp**
2	$\vdash$	$g \wedge (g \supset g_1) \supset g_1$	**Prop**
3	$\vdash$	$f ; (g \wedge (g \supset g_1)) \supset f ; g_1$	2, **RightChopImpChop**
4	$\vdash$	$f ; g \wedge f \leadsto (g \supset g_1) \supset f ; g_1$	1, 3, **ImpChain**

qed

$$\vdash \quad (f \vee f_1) \leadsto g \equiv (f \leadsto g) \wedge (f_1 \leadsto g)$$ OrYieldsImp

Proof:

1	$\vdash$	$(f \vee f_1) ; \neg g \equiv f ; \neg g \vee f_1 ; \neg g$	**OrChopEqv**
2	$\vdash$	$\neg((f \vee f) ; \neg g) \equiv \neg(f ; \neg g) \wedge \neg(f_1 ; \neg g)$	1, **Prop**
3	$\vdash$	$(f \vee f_1) \leadsto g \equiv (f \leadsto g) \wedge (f_1 \leadsto g)$	2, def. of $\leadsto$

qed

$$\vdash f \supset f_1 \quad \Rightarrow \quad \vdash (f_1 \leadsto g) \supset (f \leadsto g)$$ LeftYieldsImpYields

Proof:

1	$\vdash$	$f \supset f_1$	given
2	$\vdash$	$f ; \neg g \supset f_1 ; \neg g$	1, **LeftChopImpChop**
3	$\vdash$	$\neg(f_1 ; \neg g) \supset \neg(f ; \neg g)$	2, **Prop**
4	$\vdash$	$f_1 \leadsto g \supset f \leadsto g$	3, def. of $\leadsto$

qed

$\vdash f \equiv f_1 \quad \Rightarrow \quad \vdash (f \rightsquigarrow g) \equiv (f_1 \rightsquigarrow g)$ **LeftYieldsEqvYields**

Proof:

1	$\vdash$	$f \equiv f_1$	given
2	$\vdash$	$f \,;\, \neg g \equiv f_1 \,;\, \neg g$	1, **LeftChopEqvChop**
3	$\vdash$	$\neg(f \,;\, \neg g) \equiv \neg(f_1 \,;\, \neg g)$	2, **Prop**
4	$\vdash$	$f \rightsquigarrow g \equiv f_1 \rightsquigarrow g$	3, def. of $\rightsquigarrow$

qed

$\vdash w \wedge f \supset \text{fin } w_1 \quad \Rightarrow \quad \vdash w \supset (f \rightsquigarrow w_1)$ **StateImpYields**

Proof:

1	$\vdash$	$w \wedge f \supset \text{fin } w_1$	given
2	$\vdash$	$w \supset (f \supset \text{fin } w_1)$	1, **Prop**
3	$\vdash$	$w \supset \boxed{\text{i}}(f \supset \text{fin } w_1)$	2, **StateImpBiGen**
4	$\vdash$	$\boxed{\text{i}}(f \supset \text{fin } w_1) \equiv f \rightsquigarrow w_1$	**BiImpFinEqvYieldsState**
5	$\vdash$	$w \supset (f \rightsquigarrow w_1)$	3, 4, **EqvChain**

qed

$\vdash w \wedge f \supset f_1 \quad \Rightarrow \quad \vdash w \wedge (f_1 \rightsquigarrow g) \supset (f \rightsquigarrow g)$ **StateAndYieldsImpYields**

Proof:

1	$\vdash$	$w \wedge f \supset f_1$	given
2	$\vdash$	$w \wedge (f \,;\, \neg g) \supset f_1 \,;\, \neg g$	1, **StateAndChopImpChopRule**
3	$\vdash$	$w \wedge \neg(f_1 \,;\, \neg g) \supset \neg(f \,;\, \neg g)$	2, **Prop**
4	$\vdash$	$w \wedge (f_1 \rightsquigarrow g) \supset (f \rightsquigarrow g)$	3, def. of $\rightsquigarrow$

qed

$\vdash \quad f \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$ **AndYieldsA**

Proof:

1	$\vdash$	$f \wedge f_1 \supset f$	**Prop**
2	$\vdash$	$f \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$	1, **LeftYieldsImpYields**

qed

$\vdash \quad f_1 \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$ **AndYieldsB**

Proof:

1	$\vdash$	$f \wedge f_1 \supset f_1$	**Prop**
2	$\vdash$	$f_1 \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$	1, **LeftYieldsImpYields**

qed

$$\vdash g \supset g_1 \quad \Rightarrow \quad \vdash (f \rightsquigarrow g) \supset (f \rightsquigarrow g_1)$$ RightYieldsImpYields

Proof:

1	$\vdash$	$g \supset g_1$	given
2	$\vdash$	$\neg g_1 \supset \neg g$	1, **Prop**
3	$\vdash$	$f ; \neg g_1 \supset f ; \neg g$	2, **RightChopImpChop**
4	$\vdash$	$\neg(f ; \neg g) \supset \neg(f ; \neg g_1)$	3, **Prop**
5	$\vdash$	$(f \rightsquigarrow g) \supset (f \rightsquigarrow g_1)$	3, def. of $\rightsquigarrow$

qed

$$\vdash g \equiv g_1 \quad \Rightarrow \quad \vdash (f \rightsquigarrow g) \equiv (f \rightsquigarrow g_1)$$ RightYieldsEqvYields

Proof:

1	$\vdash$	$g \equiv g_1$	given
2	$\vdash$	$\neg g \equiv \neg g_1$	1, **Prop**
3	$\vdash$	$f ; \neg g \equiv f ; \neg g_1$	2, **RightChopEqvChop**
4	$\vdash$	$\neg(f ; \neg g) \equiv \neg(f ; \neg g_1)$	3, **Prop**
5	$\vdash$	$f \rightsquigarrow g \equiv f \rightsquigarrow g_1$	4, def. of $\rightsquigarrow$

qed

$$\vdash \quad \Box\, g \supset f \rightsquigarrow g$$ BoxImpYields

Proof:

1	$\vdash$	$f ; \neg g \supset \Diamond \neg g$	**ChopImpDiamond**
2	$\vdash$	$\neg \Diamond \neg g \supset \neg(f ; \neg g)$	1, **Prop**
3	$\vdash$	$\Box\, g \supset f \rightsquigarrow g$	2, def. of $\Box$, $\rightsquigarrow$

qed

$$\vdash \quad \Box\, f \equiv \mathsf{true} \rightsquigarrow f$$ BoxEqvTrueYields

Proof:

1	$\vdash$	$\mathsf{true} ; \neg f \equiv \Diamond \neg f$	**TrueChopEqvDiamond**
2	$\vdash$	$\neg(\mathsf{true} ; \neg f) \equiv \neg \Diamond \neg f$	1, **Prop**
3	$\vdash$	$\Box\, f \equiv \neg \Diamond \neg f$	**PTL**
4	$\vdash$	$\Box\, f \equiv \neg(\mathsf{true} ; \neg f)$	2, 3, **Prop**
5	$\vdash$	$\Box\, f \equiv \mathsf{true} \rightsquigarrow f$	4, def. of $\rightsquigarrow$

qed

$\vdash g \quad \Rightarrow \quad \vdash f \rightsquigarrow g$ YieldsGen

Proof:

1	$\vdash$	g	given
2	$\vdash$	$\Box\, g$	**BoxGen**
3	$\vdash$	$\Box\, g \supset f \rightsquigarrow g$	**BoxImpYields**
4	$\vdash$	$f \rightsquigarrow g$	2, 3, **MP**

qed

$\vdash \quad (f \rightsquigarrow g) \wedge (f \rightsquigarrow g_1) \equiv f \rightsquigarrow (g \wedge g_1)$ YieldsAndYieldsEqvYieldsAnd

Proof:

1	$\vdash$	$f\,;(\neg g \vee \neg g_1) \equiv (f\,;\neg g) \vee (f\,;\neg g_1)$	**ChopOrEqv**
2	$\vdash$	$(f\,;\neg g) \vee (f\,;\neg g_1) \equiv f\,;(\neg g \vee \neg g_1)$	1, **Prop**
3	$\vdash$	$\neg g \vee \neg g_1 \equiv \neg(g \wedge g_1)$	**Prop**
4	$\vdash$	$f\,;(\neg g \vee \neg g_1) \equiv f\,;\neg(g \wedge g_1)$	3, **RightChopEqvChop**
5	$\vdash$	$(f\,;\neg g) \vee (f\,;\neg g_1) \equiv f\,;\neg(g \wedge g_1)$	2, 4, **ImpChain**
6	$\vdash$	$\neg(f\,;\neg g) \wedge \neg(f\,;\neg g_1) \equiv \neg(f\,;\neg(g \wedge g_1))$	5, **Prop**
7	$\vdash$	$(f \rightsquigarrow g) \wedge (f \rightsquigarrow g_1) \equiv f \rightsquigarrow (g \wedge g_1)$	6, def. of $\rightsquigarrow$

qed

$\vdash \quad (f \rightsquigarrow g) \wedge (f_1 \rightsquigarrow g_1) \supset (f \wedge f_1) \rightsquigarrow (g \wedge g_1)$ YieldsAndYieldsImpAndYieldsAnd

Proof:

1	$\vdash$	$f \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$	**AndYieldsA**
2	$\vdash$	$f_1 \rightsquigarrow g_1 \supset (f \wedge f_1) \rightsquigarrow g_1$	**AndYieldsB**
3	$\vdash$	$(f \wedge f_1) \rightsquigarrow g \wedge (f \wedge f_1) \rightsquigarrow g_1 \equiv (f \wedge f_1) \rightsquigarrow (g \wedge g_1)$	**YieldsAndYieldsEqvYieldsAnd**
4	$\vdash$	$(f \rightsquigarrow g) \wedge (f_1 \rightsquigarrow g_1) \supset (f \wedge f_1) \rightsquigarrow (g \wedge g_1)$	1, 2, 3, **Prop**

qed

$\vdash \quad f \rightsquigarrow (g \rightsquigarrow h) \equiv (f\,;g) \rightsquigarrow h$ YieldsYieldsEqvChopYields

Proof:

1	$\vdash$	$(f ; g) ; \neg h \equiv f ; (g ; \neg h)$	**ChopAssoc**
2	$\vdash$	$f ; (g ; \neg h) \equiv (f ; g) ; \neg h$	1, **Prop**
3	$\vdash$	$g ; \neg h \equiv \neg\neg(g ; \neg h)$	**Prop**
4	$\vdash$	$f ; (g ; \neg h) \equiv f ; \neg\neg(g ; \neg h)$	3, **LeftChopEqvChop**
5	$\vdash$	$f ; \neg\neg(g ; \neg h) \equiv (f ; g) ; \neg h$	2, 4, **Prop**
6	$\vdash$	$f ; \neg(g \rightsquigarrow h) \equiv (f ; g) ; \neg h$	5, def. of $\rightsquigarrow$
7	$\vdash$	$\neg(f ; \neg(g \rightsquigarrow h)) \equiv \neg((f ; g) ; \neg h)$	6, **Prop**
8	$\vdash$	$f \rightsquigarrow (g \rightsquigarrow h) \equiv (f ; g) \rightsquigarrow h$	7, def. of $\rightsquigarrow$

qed

$\vdash$ empty $\rightsquigarrow f \equiv f$ — EmptyYields

Proof:

1	$\vdash$	$\text{empty} ; \neg f \equiv \neg f$	**EmptyChop**
2	$\vdash$	$\neg(\text{empty} ; \neg f) \equiv f$	1, **Prop**
3	$\vdash$	$\text{empty} \rightsquigarrow f \equiv f$	2, def. of $\rightsquigarrow$

qed

$\vdash$ $(\bigcirc f) \rightsquigarrow g \equiv \textcircled{w}(f \rightsquigarrow g)$ — NextYields

Proof:

1	$\vdash$	$(\bigcirc f) ; \neg g \equiv \bigcirc(f ; \neg g)$	**NextChop**
2	$\vdash$	$\neg((\bigcirc f) ; \neg g) \equiv \neg \bigcirc(f ; \neg g)$	1, **Prop**
3	$\vdash$	$(\bigcirc f) \rightsquigarrow g \equiv \neg \bigcirc(f ; \neg g)$	2, def. of $\rightsquigarrow$
4	$\vdash$	$\neg \bigcirc(f ; \neg g) \equiv \textcircled{w}\neg(f ; \neg g)$	**PTL**
5	$\vdash$	$(\bigcirc f) \rightsquigarrow g \equiv \textcircled{w}\neg(f ; \neg g)$	3, 4, **Prop**
6	$\vdash$	$(\bigcirc f) \rightsquigarrow g \equiv \textcircled{w}(f \rightsquigarrow g)$	5, def. of $\rightsquigarrow$

qed

$\vdash$ skip $; f \equiv \bigcirc f$ — SkipChopEqvNext

Proof:

1	$\vdash$	$(\bigcirc\text{empty}) ; f \equiv \bigcirc(\text{empty} ; f)$	**NextChop**
2	$\vdash$	$\text{empty} ; f \equiv f$	**EmptyChop**
3	$\vdash$	$\bigcirc(\text{empty} ; f) \equiv \bigcirc f$	2, **NextEqvNext**
4	$\vdash$	$(\bigcirc\text{empty}) ; f \equiv \bigcirc f$	1, 3, **Prop**
5	$\vdash$	$\text{skip} ; f \equiv \bigcirc f$	4, def. of skip

qed

$\vdash$ skip $\rightsquigarrow f \equiv \textcircled{w} f$ — SkipYieldsEqvWeakNext

Proof:

1	$\vdash$	$\mathsf{skip} ; \neg f \equiv \bigcirc\neg f$	**SkipChopEqvNext**
2	$\vdash$	$\neg(\mathsf{skip} ; \neg f) \equiv \neg\bigcirc\neg f$	1, **Prop**
3	$\vdash$	$\neg\bigcirc\neg f \equiv ⓦ f$	**PTL**
4	$\vdash$	$\neg(\mathsf{skip} ; \neg f) \equiv ⓦ f$	2, 3, **EqvChain**
5	$\vdash$	$\mathsf{skip} \leadsto f \equiv ⓦ f$	4, def. of $\leadsto$

qed

$\vdash \quad \bigcirc f \supset \mathsf{skip} \leadsto f$ NextImpSkipYields

Proof:

1	$\vdash$	$\bigcirc f \supset ⓦ f$	**PTL**
2	$\vdash$	$\mathsf{skip} \leadsto f \equiv ⓦ f$	**SkipYieldsEqvWeakNext**
3	$\vdash$	$\bigcirc f \supset \mathsf{skip} \leadsto f$	1, 2, **Prop**

qed

$\vdash \quad \mathsf{more} \equiv \mathsf{skip} ; \mathsf{true}$ MoreEqvSkipChopTrue

Proof:

1	$\vdash$	$\mathsf{skip} ; \mathsf{true} \equiv \bigcirc\mathsf{true}$	**SkipChopEqvNext**
2	$\vdash$	$\bigcirc\mathsf{true} \equiv \mathsf{skip} ; \mathsf{true}$	1, **Prop**
3	$\vdash$	$\mathsf{more} \equiv \mathsf{skip} ; \mathsf{true}$	def. of more

qed

$\vdash \quad \mathsf{more} ; f \supset \mathsf{more}$ MoreChopImpMore

Proof:

1	$\vdash$	$(\bigcirc\mathsf{true}) ; f \equiv \bigcirc(\mathsf{true} ; f)$	**NextChop**
2	$\vdash$	$\bigcirc(\mathsf{true} ; f) \supset \mathsf{more}$	**PTL**
3	$\vdash$	$(\bigcirc\mathsf{true} ; f) \supset \mathsf{more}$	1, 2, **Prop**
4	$\vdash$	$\mathsf{more} ; f \supset \mathsf{more}$	3, def. of more

qed

$\vdash \quad f ; \mathsf{more} \supset \mathsf{more}$ ChopMoreImpMore

Proof:

1	$\vdash$	$f ; \mathsf{more} \supset \Diamond\,\mathsf{more}$	**ChopImpDiamond**
2	$\vdash$	$\Diamond\,\mathsf{more} \supset \mathsf{more}$	**PTL**
3	$\vdash$	$f ; \mathsf{more} \supset \mathsf{more}$	1, 2, **ImpChain**

qed

$\vdash \quad \mathsf{more}\ ;\ f \equiv \bigcirc\Diamond\, f$	MoreChopEqvNextDiamond

Proof of $\supset$:

1	$\vdash$	$\mathsf{more}\ ;\ f \equiv (\bigcirc\mathsf{true})\ ;\ f$	def. of more
2	$\vdash$	$(\bigcirc\mathsf{true})\ ;\ f \equiv \bigcirc(\mathsf{true}\ ;\ f)$	**NextChop**
3	$\vdash$	$\mathsf{more}\ ;\ f \equiv \bigcirc(\mathsf{true}\ ;\ f)$	1, 2, **EqvChain**
4	$\vdash$	$\mathsf{more}\ ;\ f \equiv \bigcirc\Diamond$	3, def. of $\Diamond$

qed

Proof of $\subset$:

1	$\vdash$	$\Diamond\, f \supset \mathsf{true}\ ;\ f$	**DiamondImpTrueChop**
2	$\vdash$	$\bigcirc\Diamond\, f \supset \bigcirc(\mathsf{true}\ ;\ f)$	1, **NextImpNext**
3	$\vdash$	$(\bigcirc\mathsf{true})\ ;\ f \equiv \bigcirc(\mathsf{true}\ ;\ f)$	**NextChop**
4	$\vdash$	$\mathsf{more}\ ;\ f \equiv \bigcirc(\mathsf{true}\ ;\ f)$	3, def. of $\leadsto$
5	$\vdash$	$\bigcirc\Diamond\, f \supset \mathsf{more}\ ;\ f$	2, 3, 4, **Prop**

qed

The following are is an easy corollary:

$\vdash \quad \mathsf{more} \leadsto f \equiv ⓦ\Box\, f$	WeakNextBoxImpMoreYields

$\vdash \quad \neg f \equiv f \leadsto \mathsf{more}$	NotEqvYieldsMore

Proof:

1	$\vdash$	$f\ ;\ \mathsf{empty} \equiv f$	**ChopEmpty**
2	$\vdash$	$\neg(f\ ;\ \mathsf{empty}) \equiv \neg f$	1, **Prop**
3	$\vdash$	$\mathsf{empty} \equiv \neg\mathsf{more}$	def. of empty
4	$\vdash$	$f\ ;\ \mathsf{empty} \equiv f\ ;\ \neg\mathsf{more}$	3, **RightChopEqvChop**
5	$\vdash$	$\neg(f\ ;\ \mathsf{empty}) \equiv \neg(f\ ;\ \neg\mathsf{more})$	4, **Prop**
6	$\vdash$	$\neg f \equiv \neg(f\ ;\ \neg\mathsf{more})$	2, 5, **EqvChain**
7	$\vdash$	$\neg f \equiv f \leadsto \mathsf{more}$	6, def. of $\leadsto$

qed

$\vdash f \supset \mathsf{more} \quad \Rightarrow \quad \vdash f\ ;\ g \supset \mathsf{more}$	LeftChopImpMoreRule

Proof:

1	$\vdash$	$f \supset \mathsf{more}$	given
2	$\vdash$	$f\ ;\ g \supset \mathsf{more}\ ;\ g$	1, **LeftChopImpChop**
3	$\vdash$	$\mathsf{more}\ ;\ g \supset \mathsf{more}$	**MoreChopImpMore**
4	$\vdash$	$f\ ;\ g \supset \mathsf{more}$	2, 3, **ImpChain**

qed

$\vdash g \supset \text{more} \quad \Rightarrow \quad \vdash f ; g \supset \text{more}$	RightChopImpMoreRule

Proof:

1	$\vdash$	$g \supset \text{more}$	given
2	$\vdash$	$f ; g \supset f ; \text{more}$	1, **RightChopImpChop**
3	$\vdash$	$f ; \text{more} \supset \text{more}$	**MoreChopImpMore**
4	$\vdash$	$f ; g \supset \text{more}$	2, 3, **ImpChain**

qed

$\vdash \quad \neg \diamondsuit f \equiv \boxdot \neg f$	NotDiEqvBiNot

Proof:

1	$\vdash$	$f \equiv \neg\neg f$	**Prop**
2	$\vdash$	$\diamondsuit f \equiv \diamondsuit \neg\neg f$	1, **DiEqvDi**
3	$\vdash$	$\neg \diamondsuit f \equiv \neg \diamondsuit \neg\neg f$	2, **Prop**
4	$\vdash$	$\neg \diamondsuit f \equiv \boxdot \neg f$	3, def. of $\boxdot$

qed

$\vdash \quad f ; g \supset \diamondsuit f$	ChopImpDi

Proof:

1	$\vdash$	$g \supset \text{true}$	**Prop**
2	$\vdash$	$f ; g \supset f ; \text{true}$	1, **BoxChopImpChop**
3	$\vdash$	$f ; g \supset \diamondsuit f$	2, def. of $\diamondsuit$

qed

$\vdash \quad \text{true} \equiv \text{true} ; \text{true}$	TrueEqvTrueChopTrue

Proof:

1	$\vdash$	$\text{true} ; \text{true} \supset \text{true}$	**Prop**
2	$\vdash$	$\text{true} \supset \diamondsuit \text{true}$	**DiIntro**
3	$\vdash$	$\text{true} \supset \text{true} ; \text{true}$	2, def. of $\diamondsuit$
4	$\vdash$	$\text{true} \equiv \text{true} ; \text{true}$	1, 3, **Prop**

qed

$\vdash \quad \diamondsuit f \equiv \diamondsuit \diamondsuit f$	DiEqvDiDi

Proof:

1	$\vdash$	$\mathsf{true} \equiv \mathsf{true} \,;\, \mathsf{true}$	TrueEqvTrueChopTrue
2	$\vdash$	$f \,;\, \mathsf{true} \equiv f \,;\, (\mathsf{true} \,;\, \mathsf{true})$	1, RightChopEqvChop
3	$\vdash$	$(f \,;\, \mathsf{true}) \,;\, \mathsf{true} \equiv f \,;\, (\mathsf{true} \,;\, \mathsf{true})$	ChopAssoc
4	$\vdash$	$f \,;\, \mathsf{true} \equiv (f \,;\, \mathsf{true}) \,;\, \mathsf{true}$	2, 3, Prop
5	$\vdash$	$\Diamond_i f \equiv \Diamond_i \Diamond_i f$	4, def. of $\Diamond_i$

qed

$$\vdash \quad \Box_i f \equiv \Box_i \Box_i f \qquad \text{BiEqvBiBi}$$

Proof:

1	$\vdash$	$\Diamond_i \neg f \equiv \Diamond_i \Diamond_i \neg f$	DiEqvDiDi
2	$\vdash$	$\Diamond_i \neg f \equiv \neg \Box_i f$	DiNotEqvNotBi
3	$\vdash$	$\Diamond_i \Diamond_i \neg f \equiv \Diamond_i \neg \Box_i f$	2, DiEqvDi
4	$\vdash$	$\Diamond_i \neg f \equiv \Diamond_i \neg \Box_i f$	1, 3, EqvChain
5	$\vdash$	$\neg \Diamond_i \neg f \equiv \neg \Diamond_i \neg \Box_i f$	4, Prop
6	$\vdash$	$\Box_i f \equiv \Box_i \Box_i f$	5, def. of $\Box_i$

qed

$$\vdash \quad \Diamond_i (f \vee g) \equiv \Diamond_i f \vee \Diamond_i g \qquad \text{DiOrEqv}$$

Proof:

1	$\vdash$	$(f \vee g) \,;\, \mathsf{true} \equiv f \,;\, \mathsf{true} \vee g \,;\, \mathsf{true}$	OrChopEqv
2	$\vdash$	$\Diamond_i (f \vee g) \equiv \Diamond_i f \vee \Diamond_i g$	1, def. of $\Diamond_i$

qed

$$\vdash \quad \Diamond_i (f \wedge g) \supset \Diamond_i f \qquad \text{DiAndA}$$

Proof:

1	$\vdash$	$(f \wedge g) \,;\, \mathsf{true} \supset f \,;\, \mathsf{true}$	AndChopA
2	$\vdash$	$\Diamond_i (f \wedge g) \supset \Diamond_i f$	1, def. of $\Diamond_i$

qed

$$\vdash \quad \Diamond_i (f \wedge g) \supset \Diamond_i g \qquad \text{DiAndB}$$

$$\vdash \quad \Diamond_i (f \wedge g) \supset \Diamond_i f \wedge \Diamond_i g \qquad \text{DiAndImpAnd}$$

Proof:

1	$\vdash$	$\Diamond_i (f \wedge g) \supset \Diamond_i f$	DiAndA
2	$\vdash$	$\Diamond_i (f \wedge g) \supset \Diamond_i g$	DiAndB
3	$\vdash$	$\Diamond_i (f \wedge g) \supset \Diamond_i f \wedge \Diamond_i g$	1, 2, Prop

qed

$\vdash \quad \Diamond\, \mathsf{skip} \equiv \mathsf{more}$ DiSkipEqvMore

Proof:

1	$\vdash$	skip ; true $\equiv$ $\bigcirc$true	**SkipChopEqvNext**, **DiAndB**
2	$\vdash$	$\bigcirc$true $\equiv$ more	**PTL**
3	$\vdash$	skip ; true $\equiv$ more	1, 2, **Prop**
4	$\vdash$	$\Diamond$ skip $\equiv$ more	3, def. of $\Diamond$

qed

$\vdash \quad \Diamond\, \mathsf{more} \equiv \mathsf{more}$ DiMoreEqvMore

Proof for $\supset$:

1	$\vdash$	$\Diamond(\bigcirc$skip$) \equiv \bigcirc\Diamond$ true	**DiNext**
2	$\vdash$	$\bigcirc\Diamond$ skip $\supset$ more	**PTL**
3	$\vdash$	$\Diamond\bigcirc$skip $\supset$ more	1, 2, **ImpChain**
4	$\vdash$	$\Diamond$ more $\supset$ more	3, def. of more

qed

Proof of $\subset$:

1	$\vdash$	more $\supset \Diamond$ more	**ImpDi**

qed

$\vdash f \equiv \text{if } w \text{ then } g \text{ else } h \quad \Rightarrow \quad \vdash \Diamond f \equiv \text{if } w \text{ then } \Diamond g \text{ else } \Diamond h$ DiIfEqvRule

Proof:

1	$\vdash$	$f \equiv$ if w then g else h	given
2	$\vdash$	f ; true $\equiv$ if w then (g ; true) else (h ; true)	1, **IfChopEqvRule**
3	$\vdash$	$\Diamond f \equiv$ if w then $\Diamond g$ else $\Diamond h$	2, def. of $\Diamond$

qed

$\vdash \quad \Diamond\, \mathsf{empty}$ DiEmpty

Proof:

1	$\vdash$	true	**PTL**
2	$\vdash$	empty ; true $\equiv$ true	**EmptyChop**
3	$\vdash$	empty ; true	1, 2, **Prop**
4	$\vdash$	$\Diamond$ empty	3, def. of $\Diamond$

qed

2.3 Properties of Diamond-a and Box-a

$$\vdash \quad \Diamond_a f \equiv \Diamond \Diamond_i f \qquad \text{DaEqvDtDi}$$

Proof:

1	$\vdash$	$\text{true} \,;\, (f \,;\, \text{true}) \equiv \text{true} \,;\, (f \,;\, \text{true})$	**Prop**
2	$\vdash$	$\text{true} \,;\, (f \,;\, \text{true}) \equiv \text{true} \,;\, \Diamond_i f$	1, def. of $\Diamond_i$
3	$\vdash$	$\text{true} \,;\, \Diamond_i f \equiv \Diamond \Diamond_i f$	**TrueChopEqvDiamond**
4	$\vdash$	$\text{true} \,;\, f \,;\, \text{true} \equiv \Diamond \Diamond_i f$	2, 3, **EqvChain**
5	$\vdash$	$\Diamond_a f \equiv \Diamond \Diamond_i f$	4, def. of $\Diamond_a$

qed

$$\vdash \quad \Diamond_a f \equiv \Diamond_i \Diamond f \qquad \text{DaEqvDiDt}$$

Proof:

1	$\vdash$	$\text{true} \,;\, f \equiv \Diamond f$	**TrueChopEqvDiamond**
2	$\vdash$	$(\text{true} \,;\, f) \,;\, \text{true} \equiv (\Diamond f) \,;\, \text{true}$	1, **LeftChopEqvChop**
3	$\vdash$	$(\text{true} \,;\, f) \,;\, \text{true} \equiv \Diamond_i \Diamond f$	2, def. of $\Diamond_i$
4	$\vdash$	$(\text{true} \,;\, f) \,;\, \text{true} \equiv \text{true} \,;\, f \,;\, \text{true}$	**ChopAssoc**
5	$\vdash$	$\text{true} \,;\, f \,;\, \text{true} \equiv \Diamond_i \Diamond f$	3, 4, **Prop**
6	$\vdash$	$\Diamond_a f \equiv \Diamond_i \Diamond f$	5, def. of $\Diamond_a$

qed

Here is a corollary of theorems **DaEqvDtDi** and **DaEqvDiDt**:

$$\vdash \quad \Diamond \Diamond_i f \equiv \Diamond_i \Diamond f \qquad \text{DtDiEqvDiDt}$$

$$\vdash \quad \boxed{a}\, f \equiv \boxed{i}\, \Box f \qquad \text{BaEqvBiBt}$$

Proof:

1	$\vdash$	$\Diamond_a \neg f \equiv \Diamond_i \Diamond \neg f$	**DaEqvDiDt**
2	$\vdash$	$\Diamond \neg f \equiv \neg \Box f$	**PTL**
3	$\vdash$	$\Diamond_i \Diamond \neg f \equiv \Diamond_i \neg \Box f$	2, **DiEqvDi**
4	$\vdash$	$\Diamond_a \neg f \equiv \Diamond_i \neg \Box f$	1, 3, **EqvChain**
5	$\vdash$	$\neg \Diamond_a \neg f \equiv \neg \Diamond_i \neg \Box f$	4, **Prop**
6	$\vdash$	$\neg \Diamond_a \neg f \equiv \boxed{i}\, \Box f$	5, def. of $\boxed{i}$
7	$\vdash$	$\boxed{a}\, f \equiv \boxed{i}\, \Box f$	6, def. of $\boxed{a}$

qed

$$\vdash \quad \boxed{a}\, f \equiv \Box \boxed{i}\, f \qquad \text{BaEqvBtBi}$$

Proof:
1 $\vdash$ $\Diamond_{a} \neg f \equiv \Diamond \Diamond_{i} \neg f$ DaEqvDtDi
2 $\vdash$ $\Diamond_{i} \neg f \equiv \neg \boxed{i}\, f$ DiNotEqvNotBi
3 $\vdash$ $\Diamond \Diamond_{i} \neg f \equiv \Diamond \neg \boxed{i}\, f$ 2, DiamondEqvDiamond
4 $\vdash$ $\neg \Diamond \neg \boxed{i}\, f \equiv \Box \boxed{i}\, f$ PTL
5 $\vdash$ $\neg \Diamond_{a} \neg f \equiv \Box \boxed{i}\, f$ 1, 3, 4, Prop
6 $\vdash$ $\boxed{a}\, f \equiv \Box \boxed{i}\, f$ 5, def. of $\boxed{a}$
qed

The following is a corollary of theorems BaEqvBtBi and BaEqvBiBt:

$\vdash$ $\Box \boxed{i}\, f \equiv \boxed{i} \Box f$ BtBiEqvBiBt

$\vdash$ $\Diamond_{a} \neg f \equiv \neg \boxed{a}\, f$ DaNotEqvNotBa

Proof:
1 $\vdash$ $\boxed{a}\, f \equiv \neg \Diamond_{a} \neg f$ def. of $\boxed{a}$
2 $\vdash$ $\Diamond_{a} \neg f \equiv \neg \boxed{a}\, f$ 1, Prop
qed

$\vdash$ $\Diamond_{a} f \equiv \neg \boxed{a} \neg f$ DaEqvNotBaNot

Proof:
1 $\vdash$ $\boxed{a} \neg f \equiv \neg \Diamond_{a} \neg\neg f$ def. of $\boxed{a}$
2 $\vdash$ $\Diamond_{a} \neg\neg f \equiv \neg \boxed{a} \neg f$ 1, Prop
3 $\vdash$ $f \equiv \neg\neg f$ Prop
4 $\vdash$ $\Diamond_{a} f \equiv \Diamond_{a} \neg\neg f$ 3, DaEqvDa
5 $\vdash$ $\Diamond_{a} f \equiv \neg \boxed{a} \neg f$ 2, 4, EqvChain
qed

$\vdash$ $\boxed{a}\, f \supset f$ BaElim

Proof:
1 $\vdash$ $\boxed{a}\, f \equiv \Box \boxed{i}\, f$ BaEqvBtBi
1 $\vdash$ $\boxed{i}\, f \supset f$ BiElim
2 $\vdash$ $\Box(\boxed{i}\, f \supset f)$ 1, BoxGen
3 $\vdash$ $\Box(\boxed{i}\, f \supset f) \supset \Box \boxed{i}\, f \supset \Box f$ PTL
4 $\vdash$ $\Box \boxed{i}\, f \supset \Box f$ 2, 3, MP
5 $\vdash$ $\Box f \supset f$ PTL
6 $\vdash$ $\boxed{a}\, f \supset f$ 1, 4, 5, Prop

qed

Here is a corollary:

$\vdash \quad f \supset \Diamond\!\!\!\!\!{\scriptstyle a}\;\, f$ **DaIntro**

Proof:

1	$\vdash$	$\boxed{a} \neg f \supset \neg f$	**BaElim**
2	$\vdash$	$\neg\neg f \supset \neg \boxed{a} \neg f$	1, **Prop**
3	$\vdash$	$f \equiv \neg\neg f$	**Prop**
4	$\vdash$	$\Diamond\!\!\!\!\!{\scriptstyle a}\;\, f \equiv \neg \boxed{a} \neg f$	**DaEqvNotBaNot**
5	$\vdash$	$f \supset \Diamond\!\!\!\!\!{\scriptstyle a}\;\, f$	2, 3, 4, **Prop**

qed

$\vdash \quad \boxed{a} f \supset \Box f$ **BaImpBt**

Proof:

1	$\vdash$	$\boxed{a} f \equiv \boxed{i} \Box f$	**BaEqvBiBt**
2	$\vdash$	$\boxed{i} \Box f \supset \Box f$	**BiElim**
3	$\vdash$	$\boxed{a} f \supset \Box f$	1, 2, **MP**

qed

Here is an easy corollary:

$\vdash \quad \Diamond f \supset \Diamond\!\!\!\!\!{\scriptstyle a}\;\, f$ **DiamondImpDa**

$\vdash \quad \boxed{a} f \supset \boxed{i} f$ **BaImpBi**

Proof:

1	$\vdash$	$\boxed{a} f \equiv \Box \boxed{i} f$	**BaEqvBtBi**
2	$\vdash$	$\Box \boxed{i} f \supset \boxed{i} f$	**PTL**
3	$\vdash$	$\boxed{a} f \supset \boxed{i} f$	1, 2, **MP**

qed

Here is an easy corollary:

$\vdash \quad \Diamond\!\!\!\!\!{\scriptstyle i}\;\, f \supset \Diamond\!\!\!\!\!{\scriptstyle a}\;\, f$ **DiImpDa**

$\vdash f \quad \Rightarrow \quad \vdash \boxed{a} f$ **BaGen**

Proof:

1	$\vdash$	f	given
2	$\vdash$	$\Box f$	1, BoxGen
3	$\vdash$	$\boxed{i}\, \Box f$	2, BiGen
4	$\vdash$	$\boxed{a}\, f \equiv \boxed{i}\, \Box f$	BaEqvBiBt
5	$\vdash$	$\boxed{i}\, \Box f \supset \boxed{a}\, f$	4, Prop
6	$\vdash$	$\boxed{a}\, f$	3, 5, MP

qed

$\vdash \quad \boxed{a}(f \supset g) \supset \boxed{a}\, f \supset \boxed{a}\, g$ BaImpDist

Proof:

1	$\vdash$	$\boxed{i}(f \supset g) \supset (\boxed{i}\, f \supset \boxed{i}\, g)$	BiImpDist
2	$\vdash$	$\Box(\boxed{i}(f \supset g) \supset (\boxed{i}\, f \supset \boxed{i}\, g))$	BoxGen
3	$\vdash$	$\Box(\boxed{i}(f \supset g) \supset (\boxed{i}\, f \supset \boxed{i}\, g)) \supset$ $(\Box\,\boxed{i}(f \supset g) \supset (\Box\,\boxed{i}\, f \supset \Box\,\boxed{i}\, g))$	PTL
4	$\vdash$	$\Box\,\boxed{i}(f \supset g) \supset (\Box\,\boxed{i}\, f \supset \Box\,\boxed{i}\, g)$	2, 3, MP
5	$\vdash$	$\boxed{a}(f \supset g) \equiv \Box\,\boxed{i}(f \supset g)$	BaEqvBtBi
6	$\vdash$	$\boxed{a}\, f \equiv \Box\,\boxed{i}\, f$	BaEqvBtBi
7	$\vdash$	$\boxed{a}\, g \equiv \Box\,\boxed{i}\, g$	BaEqvBtBi
8	$\vdash$	$\boxed{a}(f \supset g) \supset (\boxed{a}\, f \supset \boxed{a}\, g)$	4, 5, 6, 7, Prop

qed

Here are some easy corollaries:

$\vdash \quad \boxed{a}(f \equiv g) \supset \boxed{a}\, f \equiv \boxed{a}\, g$ BaImpBaEqvBa

$\vdash f \supset g \quad \Rightarrow \quad \boxed{a}\, f \supset \boxed{a}\, g$ BaImpBa

$\vdash f \equiv g \quad \Rightarrow \quad \boxed{a}\, f \equiv \boxed{a}\, g$ BaEqvBa

$\vdash f \supset g \quad \Rightarrow \quad \Diamond_a\, f \supset \Diamond_a\, g$ DaImpDa

$\vdash f \equiv g \quad \Rightarrow \quad \Diamond_a\, f \equiv \Diamond_a\, g$ DaEqvDa

$\vdash \quad \boxed{a}(f \wedge g) \equiv \boxed{a}\, f \wedge \boxed{a}\, g$ BaAndEqv

$\vdash \quad \boxed{a}(f_1 \wedge \ldots \wedge f_n) \equiv \boxed{a}\, f_1 \wedge \ldots \wedge \boxed{a}\, f_n$ BaAndGroupEqv

$\vdash \quad \Diamond_a f \equiv \Diamond_a \Diamond_a f$ **DaEqvDaDa**

Proof:

1	$\vdash$	$\Diamond_a f \equiv \Diamond \Diamond_i f$	**DaEqvDtDi**
2	$\vdash$	$\Diamond_i f \equiv \Diamond_i \Diamond_i f$	**DiEqvDiDi**
3	$\vdash$	$\Diamond \Diamond_i f \equiv \Diamond \Diamond_i \Diamond_i f$	2, **DiamondImpDiamond**
4	$\vdash$	$\Diamond \Diamond_i f \equiv \Diamond \Diamond \Diamond_i \Diamond_i f$	**PTL**
5	$\vdash$	$\Diamond \Diamond_i \Diamond_i f \equiv \Diamond_i \Diamond \Diamond_i f$	**DtDiEqvDiDt**
6	$\vdash$	$\Diamond \Diamond \Diamond_i \Diamond_i f \equiv \Diamond \Diamond_i \Diamond \Diamond_i f$	5, **DiamondEqvDiamond**
7	$\vdash$	$\Diamond_a f \equiv \Diamond \Diamond_i \Diamond \Diamond_i f$	1, 3, 4, 6, **EqvChain**
8	$\vdash$	$\Diamond_a \Diamond \Diamond_i f \equiv \Diamond \Diamond_i \Diamond \Diamond_i f$	**DaEqvDtDi**
9	$\vdash$	$\Diamond_a \Diamond_a f \equiv \Diamond_a \Diamond \Diamond_i f$	1, **DaEqvDa**
10	$\vdash$	$\Diamond_a f \equiv \Diamond_a \Diamond_a f$	7, 8, 9, **Prop**

qed

$\vdash \quad \Box_a f \equiv \Box_a \Box_a f$ **BaEqvBaBa**

Proof:

1	$\vdash$	$\Diamond_a \neg f \equiv \Diamond_a \Diamond_a \neg f$	**DaEqvDaDa**
2	$\vdash$	$\Diamond_a \Diamond_a \neg f \equiv \neg \Box_a \neg \Diamond_a \neg f$	**DaEqvNotBaNot**
3	$\vdash$	$\neg \Diamond_a \Diamond_a \neg f \equiv \Box_a \neg \Diamond_a \neg f$	2, **Prop**
4	$\vdash$	$\neg \Diamond_a \neg f \equiv \Box_a \neg \Diamond_a \neg f$	1, 3, **Prop**
5	$\vdash$	$\Box_a f \equiv \Box_a \Box_a f$	4, def. of $\Box_a$

qed

$\vdash \quad \Box_a(f \supset f_1) \supset f ; g \supset f ; g_1$ **BaLeftChopImpChop**

Proof:

1	$\vdash$	$\Box_a(f \supset f_1) \supset \Box_i(f \supset f_1)$	**BaImpBi**
2	$\vdash$	$\Box_i(f \supset f_1) \supset f ; g \supset f_1 ; g$	**BiChopImpChop**
3	$\vdash$	$\Box_a(f \supset f_1) \supset f ; g \supset f_1 ; g$	1, 2, **Prop**

qed

$\vdash \quad \Box_a(g \supset g_1) \supset f ; g \supset f ; g_1$ **BaRightChopImpChop**

Proof:

1	$\vdash$	$\Box_a(g \supset g_1) \supset \Box(g \supset g_1)$	**BaImpBt**
2	$\vdash$	$\Box(g \supset g_1) \supset f ; g \supset f ; g_1$	**BoxChopImpChop**
3	$\vdash$	$\Box_a(g \supset g_1) \supset f ; g \supset f ; g_1$	1, 2, **Prop**

qed

$\vdash \quad \boxed{a}\, f \wedge (g\, ;\, g_1) \supset (f \wedge g)\, ;\, (f \wedge g_1)$ **BaAndChopImport**

Proof:

1	$\vdash$	$\boxed{a}\, f \supset \boxed{i}\, f$	**BaImpBi**
2	$\vdash$	$\boxed{i}\, f \wedge (g\, ;\, g_1) \supset (f \wedge g)\, ;\, g_1$	**BiAndChopImport**
3	$\vdash$	$\boxed{a}\, f \supset \Box\, f$	**BaImpBt**
4	$\vdash$	$\Box\, f \wedge (f \wedge g)\, ;\, g_1 \supset (f \wedge g)\, ;\, (f \wedge g_1)$	**BoxAndChopImport**
5	$\vdash$	$\boxed{a}\, f \wedge (g\, ;\, g_1) \supset (f \wedge g)\, ;\, (f \wedge g_1)$	1, 2, 3, 4, **Prop**

qed

$\vdash \quad (f\, ;\, f_1) \wedge \boxed{a}\, g \supset (f \wedge g)\, ;\, (f_1 \wedge g)$ **ChopAndBaImport**

Proof:

1	$\vdash$	$\boxed{a}\, g \wedge (f\, ;\, f_1) \supset (g \wedge f)\, ;\, (g \wedge f_1)$	**BaAndChopImport**
2	$\vdash$	$(g \wedge f)\, ;\, (g \wedge f_1) \equiv (f \wedge g)\, ;\, (f_1 \wedge g)$	**AndChopAndCommute**
2	$\vdash$	$(f\, ;\, f_1) \wedge \boxed{a}\, g \supset (f \wedge g)\, ;\, (f_1 \wedge g)$	1, 2, **Prop**

qed

$\vdash \quad \boxed{a}\, f \supset g\, ;\, g_1 \;\supset\; g\, ;\, (\boxed{a}\, f \wedge g_1)$ **BaChopImpChopBa**

Proof:

1	$\vdash$	$\boxed{a}\, f \supset \boxed{a}(g_1 \;\supset\; \boxed{a}\, f \wedge g_1)$	**PTL**
2	$\vdash$	$\boxed{a}(g_1 \;\supset\; \boxed{a}\, f \wedge g_1) \supset g\, ;\, g_1 \;\supset\; g\, ;\, (\boxed{a}\, f \wedge g_1)$	**BaRightChopImpChop**
3	$\vdash$	$\boxed{a}\, f \supset g\, ;\, g_1 \;\supset\; g\, ;\, (\boxed{a}\, f \wedge g_1)$	1, 2, **Prop**

qed

$\vdash \quad \Box\, w \equiv \boxed{a}\, \Box\, w$ **BoxStateEqvBaBoxState**

Proof:

1	$\vdash$	$w \equiv \boxed{i}\, w$	**StateEqvBi**
2	$\vdash$	$\Box\, w \equiv \Box\, \boxed{i}\, w$	1, **BoxEqvBox**
3	$\vdash$	$\Box\, \boxed{i}\, w \equiv \boxed{i}\, \Box\, w$	**BtBiEqvBiBt**
4	$\vdash$	$\Box\, w \equiv \Box\, \Box\, w$	**PTL**
5	$\vdash$	$\boxed{i}\, \Box\, w \equiv \boxed{i}\, \Box\, \Box\, w$	4, **BiEqvBi**
6	$\vdash$	$\boxed{a}\, \Box\, w \equiv \boxed{i}\, \Box\, \Box\, w$	**BaEqvBiBt**
7	$\vdash$	$\Box\, w \equiv \boxed{a}\, \Box\, w$	2, 3, 5, 6, **Prop**

qed

$\vdash \quad \Diamond_i\, \neg\, \boxed{a}\, f \supset \neg\, \boxed{a}\, f$ **DiNotBaImpNotBa**

Proof:
1 $\vdash$ $\boxed{a}\, f \equiv \boxed{a}\,\boxed{a}\, f$ **BaEqvBaBa**
2 $\vdash$ $\boxed{a}\,\boxed{a}\, f \supset \boxed{i}\,\boxed{a}\, f$ **BaImpBi**
3 $\vdash$ $\boxed{a}\, f \supset \boxed{i}\,\boxed{a}\, f$ 1, 2, **Prop**
4 $\vdash$ $\boxed{a}\, f \supset \neg \Diamond\!\!\!\!\!{\scriptstyle i}\;\, \neg \boxed{a}\, f$ 3, def. of $\boxed{a}$
5 $\vdash$ $\Diamond\!\!\!\!\!{\scriptstyle i}\;\, \neg \boxed{a}\, f \supset \neg \boxed{a}\, f$ 4, **Prop**
qed

$\vdash \quad (\neg \boxed{a}\, f)\,;\, g \supset \neg \boxed{a}\, f$ NotBaChopImpNotBa

Proof:
1 $\vdash$ $(\neg \boxed{a}\, f)\,;\, g \supset \Diamond\!\!\!\!\!{\scriptstyle i}\;\, \neg \boxed{a}\, f$ **ChopImpDi**
2 $\vdash$ $\Diamond\!\!\!\!\!{\scriptstyle i}\;\, \neg \boxed{a}\, f \supset \neg \boxed{a}\, f$ **DiNotBaImpNotBa**
3 $\vdash$ $(\neg \boxed{a}\, f)\,;\, g \supset \neg \boxed{a}\, f$ 1, 2, **ImpChain**
qed

2.4 Properties of Fin

$\vdash \quad f \wedge \mathrm{fin}\ w \equiv f\,;\,(w \wedge \mathrm{empty})$ AndFinEqvChopStateAndEmpty

Proof for $\supset$:
1 $\vdash$ $f\,;\,\mathrm{empty} \equiv f$ **ChopEmpty**
2 $\vdash$ $\mathrm{empty} \supset (w \wedge \mathrm{empty}) \vee (\neg w \wedge \mathrm{empty})$ **Prop**
3 $\vdash$ $f\,;\,\mathrm{empty} \supset f\,;\,(w \wedge \mathrm{empty}) \vee f\,;\,(\neg w \wedge \mathrm{empty})$ 2, **RightChopImpChop**
4 $\vdash$ $f\,;\,(\neg w \wedge \mathrm{empty}) \supset \Diamond(\neg w \wedge \mathrm{empty})$ **ChopImpDiamond**
5 $\vdash$ $\Diamond(\neg w \wedge \mathrm{empty}) \supset \neg\, \mathrm{fin}\ w$ **PTL**
6 $\vdash$ $f \supset f\,;\,(w \wedge \mathrm{empty}) \vee \neg\, \mathrm{fin}\ w$ 1, 3, 4, 5, **Prop**
7 $\vdash$ $f \wedge \mathrm{fin}\ w \supset f\,;\,(w \wedge \mathrm{empty})$ 6, **Prop**
qed

Proof for $\subset$:
1 $\vdash$ $f\,;\,(w \wedge \mathrm{empty}) \supset f\,;\,\mathrm{empty}$ **ChopAndB**
2 $\vdash$ $f\,;\,\mathrm{empty} \equiv f$ **ChopEmpty**
3 $\vdash$ $f\,;\,(w \wedge \mathrm{empty}) \supset \Diamond(w \wedge \mathrm{empty})$ **ChopImpDiamond**
4 $\vdash$ $\Diamond(w \wedge \mathrm{empty}) \supset \mathrm{fin}\ w$ **PTL**
5 $\vdash$ $f\,;\,(w \wedge \mathrm{empty}) \supset f \wedge \mathrm{fin}\ w$ 1, 2, 3, 4, **Prop**
qed

The following is a lemma used in the proof of theorem **FinChopEqvDiamond**.

$\vdash \quad (\mathrm{fin}\ w)\,;\, f \equiv (w \wedge f) \vee \bigcirc((\mathrm{fin}\ w)\,;\, f)$ FinChopEqvOr

Proof:

1	$\vdash$	$\mathsf{fin}\ w \equiv (w \wedge \mathsf{empty}) \vee \bigcirc \mathsf{fin}\ w$	**PTL**
2	$\vdash$	$(\mathsf{fin}\ w) ; f \equiv ((w \wedge \mathsf{empty}) \vee \bigcirc \mathsf{fin}\ w) ; f$	1, **LeftChopEqvChop**
3	$\vdash$	$((w \wedge \mathsf{empty}) \vee \bigcirc \mathsf{fin}\ w) ; f \equiv (w \wedge \mathsf{empty}) ; f \vee (\bigcirc \mathsf{fin}\ w) ; f$	**OrChopEqv**
4	$\vdash$	$(w \wedge \mathsf{empty}) ; f \equiv w \wedge f$	**StateAndEmptyChop**
5	$\vdash$	$(\bigcirc \mathsf{fin}\ w) ; f \equiv \bigcirc((\mathsf{fin}\ w) ; f)$	**NextChop**
6	$\vdash$	$(\mathsf{fin}\ w) ; f \equiv (w \wedge f) \vee \bigcirc((\mathsf{fin}\ w) ; f)$	2, 4, 5, **Prop**

qed

$\vdash \quad (\mathsf{fin}\ w) ; f \equiv \Diamond(w \wedge f)$ **FinChopEqvDiamond**

Proof for $\supset$:

1	$\vdash$	$(\mathsf{fin}\ w) ; f \equiv (w \wedge f) \vee \bigcirc((\mathsf{fin}\ w) ; f)$	**FinChopEqvOr**
2	$\vdash$	$\Diamond(w \wedge f) \equiv (w \wedge f) \vee \bigcirc \Diamond(w \wedge f)$	**PTL**
3	$\vdash$	$(\mathsf{fin}\ w) ; f \wedge \neg \Diamond(w \wedge f) \equiv \bigcirc((\mathsf{fin}\ w) ; f) \wedge \neg \bigcirc \Diamond(w \wedge f)$	1, 2, **Prop**
4	$\vdash$	$(\mathsf{fin}\ w) ; f \supset \Diamond(w \wedge f)$	3, **NextContra**

qed

Proof of $\subset$:

1	$\vdash$	$(\mathsf{fin}\ w) ; f \equiv (w \wedge f) \vee \bigcirc((\mathsf{fin}\ w) ; f)$	**FinChopEqvOr**
2	$\vdash$	$\Diamond(w \wedge f) \equiv (w \wedge f) \vee \bigcirc \Diamond(w \wedge f)$	**PTL**
3	$\vdash$	$\Diamond(w \wedge f) \wedge \neg((\mathsf{fin}\ w) ; f) \equiv \bigcirc \Diamond(w \wedge f) \wedge \neg \bigcirc((\mathsf{fin}\ w) ; f)$	1, 2, **Prop**
4	$\vdash$	$\Diamond(w \wedge f) \supset (\mathsf{fin}\ w) ; f$	3, **NextContra**

qed

The following important theorem demonstrates how to pass information from the final state of a subinterval to the starting state of a subinterval immediately following it.

$\vdash \quad (\mathsf{fin}\ w) \rightsquigarrow w$ **FinYields**

Proof:

1	$\vdash$	$(\mathsf{fin}\ w) ; \neg w \equiv \Diamond(w \wedge \neg w)$	**FinChopEqvDiamond**
2	$\vdash$	$\neg \Diamond(w \wedge \neg w)$	**PTL**
3	$\vdash$	$\neg((\mathsf{fin}\ w) ; \neg w)$	1, 2, **Prop**
4	$\vdash$	$(\mathsf{fin}\ w) \rightsquigarrow w$	3, def. of $\rightsquigarrow$

qed

Here is a related theorem whose proof uses **FinYields**:

$\vdash \quad (f \wedge \mathsf{fin}\ w) ; g \equiv f ; (w \wedge g)$ **AndFinChopEqvStateAndChop**

Proof for $\supset$:

1	$\vdash$	$(\mathrm{fin}\ w) \rightsquigarrow w$	**FinYields**
2	$\vdash$	$f \wedge \mathrm{fin}\ w \supset \mathrm{fin}\ w$	**Prop**
3	$\vdash$	$(\mathrm{fin}\ w) \rightsquigarrow w \supset (f \wedge \mathrm{fin}\ w) \rightsquigarrow w$	2, **LeftYieldsImpYields**
4	$\vdash$	$(f \wedge \mathrm{fin}\ w) \rightsquigarrow w$	1, 3, **MP**
5	$\vdash$	$(f \wedge \mathrm{fin}\ w) ; g \wedge (f \wedge \mathrm{fin}\ w) \rightsquigarrow w \supset (f \wedge \mathrm{fin}\ w) ; (g \wedge w)$	**ChopAndYieldsImp**
6	$\vdash$	$(f \wedge \mathrm{fin}\ w) ; g \supset (f \wedge \mathrm{fin}\ w) ; (g \wedge w)$	4, 5, **Prop**
7	$\vdash$	$(f \wedge \mathrm{fin}\ w) ; (g \wedge w) \supset f ; (g \wedge w)$	**AndChopB**
8	$\vdash$	$g \wedge w \supset w \wedge g$	**Prop**
9	$\vdash$	$f ; (g \wedge w) \supset f ; (w \wedge g)$	8, **LeftChopImpChop**
10	$\vdash$	$(f \wedge \mathrm{fin}\ w) ; g \supset f ; (w \wedge g)$	6, 7, 9, **ImpChain**

qed

Proof of $\subset$:

1	$\vdash$	$f \supset (f \wedge \mathrm{fin}\ w) \vee \mathrm{fin}\ \neg w$	**PTL**
2	$\vdash$	$f ; (w \wedge g) \supset ((f \wedge \mathrm{fin}\ w) \vee \mathrm{fin}\ \neg w) ; (w \wedge g)$	1, **LeftChopImpChop**
3	$\vdash$	$((f \wedge \mathrm{fin}\ w) \vee \mathrm{fin}\ \neg w) ; (w \wedge g) \equiv$ $(f \wedge \mathrm{fin}\ w) ; (w \wedge g) \vee (\mathrm{fin}\ \neg w) ; (w \wedge g)$	**OrChopEqv**
4	$\vdash$	$(\mathrm{fin}\ \neg w) ; (w \wedge g) \supset \Diamond(\neg w \wedge (w \wedge g))$	**FinChopEqvDiamond**
5	$\vdash$	$\neg \Diamond(\neg w \wedge (w \wedge g))$	**PTL**
6	$\vdash$	$f ; (w \wedge g) \supset (f \wedge \mathrm{fin}\ w) ; (w \wedge g)$	2, 3, 4, 5, **Prop**
7	$\vdash$	$(f \wedge \mathrm{fin}\ w) ; (w \wedge g) \supset (f \wedge \mathrm{fin}\ w) ; g$	**ChopAndB**
8	$\vdash$	$f ; (w \wedge g) \supset (f \wedge \mathrm{fin}\ w) ; g$	6, 7, **ImpChain**

qed

$\vdash \quad \diamonddot(f \wedge \mathrm{fin}\ w) \equiv f ; w$ **DiAndFinEqvChopState**

Proof:

1	$\vdash$	$(f \wedge \mathrm{fin}\ w) ; \mathrm{true} \equiv f ; (w \wedge \mathrm{true})$	**AndFinChopEqvStateAndChop**
2	$\vdash$	$w \wedge \mathrm{true} \equiv w$	**Prop**
3	$\vdash$	$f ; (w \wedge \mathrm{true}) \equiv f ; w$	2, **RightChopEqvChop**
4	$\vdash$	$(f \wedge \mathrm{fin}\ w) ; \mathrm{true} \equiv f ; w$	1, 3, **EqvChain**
5	$\vdash$	$\diamonddot(f \wedge \mathrm{fin}\ w) \equiv f ; w$	4, def. of $\diamonddot$

qed

Here is a corollary of **DiAndFinEqvChopState**:

$\vdash \quad \boxed{i}(f \supset \mathrm{fin}\ w) \equiv f \rightsquigarrow w$ **BiImpFinEqvYieldsState**

Proof:

1	$\vdash$	$\diamondsuit\!\!\!\!\cdot\ (f \wedge \mathsf{fin}\ \neg w) \equiv f\ ;\ \neg w$	DiAndFinEqvChopState
2	$\vdash$	$f \wedge \mathsf{fin}\ \neg w \equiv f \wedge \neg\, \mathsf{fin}\ w$	PTL
3	$\vdash$	$f \wedge \neg\, \mathsf{fin}\ w \equiv \neg(f \supset \mathsf{fin}\ w)$	Prop
4	$\vdash$	$f \wedge \mathsf{fin}\ \neg w \equiv \neg(f \supset \mathsf{fin}\ w)$	2, 3, EqvChain
5	$\vdash$	$\diamondsuit\!\!\!\!\cdot\ (f \wedge \mathsf{fin}\ \neg w) \equiv \diamondsuit\!\!\!\!\cdot\ \neg(f \supset \mathsf{fin}\ w)$	4, DiEqvDi
6	$\vdash$	$\diamondsuit\!\!\!\!\cdot\ \neg(f \supset \mathsf{fin}\ w) \equiv \neg\, \boxdot(f \supset \mathsf{fin}\ w)$	DiNotEqvNotBi
7	$\vdash$	$\neg\, \boxdot(f \supset \mathsf{fin}\ w) \equiv f\ ;\ \neg w$	1, 5, 6, Prop
8	$\vdash$	$\boxdot(f \supset \mathsf{fin}\ w) \equiv \neg(f\ ;\ \neg w)$	7, Prop
9	$\vdash$	$\boxdot(f \supset \mathsf{fin}\ w) \equiv f \rightsquigarrow w$	8, def. of $\rightsquigarrow$

qed

$\vdash\ f\ ;\ \mathsf{fin}\ w \supset \mathsf{fin}\ w$ ChopFinImpFin

Proof:

1	$\vdash$	$f\ ;\ \mathsf{fin}\ w \supset \diamondsuit\, \mathsf{fin}\ w$	ChopImpDiamond
2	$\vdash$	$\diamondsuit\, \mathsf{fin}\ w \supset \mathsf{fin}\ w$	PTL
3	$\vdash$	$f\ ;\ \mathsf{fin}\ w \supset \mathsf{fin}\ w$	1, 2, ImpChain

qed

$\vdash\ \mathsf{fin}\ w \supset f \rightsquigarrow \mathsf{fin}\ w$ FinImpYieldsFin

Proof:

1	$\vdash$	$f\ ;\ \mathsf{fin}\ \neg w \supset \mathsf{fin}\ \neg w$	ChopFinImpFin
2	$\vdash$	$\mathsf{fin}\ \neg w \equiv \neg\, \mathsf{fin}\ w$	PTL
3	$\vdash$	$f\ ;\ \mathsf{fin}\ \neg w \equiv f\ ;\ \neg\, \mathsf{fin}\ w$	2, RightChopEqvChop
4	$\vdash$	$f\ ;\ \neg\, \mathsf{fin}\ w \supset \neg\, \mathsf{fin}\ w$	1, 2, 3, Prop
5	$\vdash$	$\mathsf{fin}\ w \supset \neg(f\ ;\ \neg\, \mathsf{fin}\ w)$	4, Prop
6	$\vdash$	$\mathsf{fin}\ w \supset f \rightsquigarrow \mathsf{fin}\ w$	5, def. of $\rightsquigarrow$

qed

$\vdash\ (f\ ;\ g) \wedge \mathsf{fin}\ w \equiv f\ ;\ (g \wedge \mathsf{fin}\ w)$ ChopAndFin

Proof for $\supset$:

1	$\vdash$	$\mathsf{fin}\ w \supset \mathsf{true} \rightsquigarrow \mathsf{fin}\ w$	FinImpYieldsFin
2	$\vdash$	$(f\ ;\ g) \wedge \mathsf{fin}\ w \supset (f\ ;\ g) \wedge \mathsf{true} \rightsquigarrow \mathsf{fin}\ w$	1, Prop
3	$\vdash$	$(f\ ;\ g) \wedge \mathsf{true} \rightsquigarrow \mathsf{fin}\ w \supset f\ ;\ (g \wedge \mathsf{fin}\ w)$	ChopAndYieldsImp
4	$\vdash$	$(f\ ;\ g) \wedge \mathsf{fin}\ w \supset f\ ;\ (g \wedge \mathsf{fin}\ w)$	2, 3, ImpChain

qed

Proof for $\subset$:

1	$\vdash$	$f ; (g \wedge \mathsf{fin}\ w) \supset f ; g$	**ChopAndA**
2	$\vdash$	$f ; (g \wedge \mathsf{fin}\ w) \supset f ; \mathsf{fin}\ w$	**ChopAndB**
3	$\vdash$	$f ; \mathsf{fin}\ w \supset \Diamond \mathsf{fin}\ w$	**ChopImpDiamond**
4	$\vdash$	$\Diamond \mathsf{fin}\ w \supset \mathsf{fin}\ w$	**PTL**
5	$\vdash$	$f ; (g \wedge \mathsf{fin}\ w) \supset (f ; g) \wedge \mathsf{fin}\ w$	1, 2, 3, 4, **Prop**

qed

Here is a corollary used in some proofs by contradiction:

$$\vdash \quad f ; g \wedge \neg \mathsf{fin}\ w \equiv f ; (g \wedge \neg \mathsf{fin}\ w)$$ ChopAndNotFin

Proof:

1	$\vdash$	$f ; g \wedge \mathsf{fin}\ \neg w \equiv f ; (g \wedge \mathsf{fin}\ \neg w)$	**ChopAndFin**
2	$\vdash$	$\mathsf{fin}\ \neg w \equiv \neg \mathsf{fin}\ w$	**PTL**
3	$\vdash$	$g \wedge \mathsf{fin}\ \neg w \equiv g \wedge \neg \mathsf{fin}\ w$	2, **Prop**
4	$\vdash$	$f ; (g \wedge \mathsf{fin}\ \neg w) \equiv f ; (g \wedge \neg \mathsf{fin}\ w)$	3, **LeftChopEqvChop**
5	$\vdash$	$f ; g \wedge \neg \mathsf{fin}\ w \equiv f ; (g \wedge \neg \mathsf{fin}\ w)$	1, 2, 4, **Prop**

qed

$$\vdash \quad (w \supset \mathsf{fin}\ w_1) ; (w_1 \supset \mathsf{fin}\ w_2) \supset (w \supset \mathsf{fin}\ w_2)$$ FinChopChain

Proof:

1	$\vdash$	$w \wedge (w \supset \mathsf{fin}\ w_1) ; (w_1 \supset \mathsf{fin}\ w_2) \supset$ $(w \wedge (w \supset \mathsf{fin}\ w_1)) ; (w_1 \supset \mathsf{fin}\ w_2))$	**StateAndChopImport**
2	$\vdash$	$w \wedge (w \supset \mathsf{fin}\ w_1) \supset \mathsf{fin}\ w_1$	**Prop**
3	$\vdash$	$(w \wedge (w \supset \mathsf{fin}\ w_1)) ; (w_1 \supset \mathsf{fin}\ w_2) \supset$ $(\mathsf{fin}\ w_1) ; (w_1 \supset \mathsf{fin}\ w_2)$	2, **LeftChopImpChop**
4	$\vdash$	$(\mathsf{fin}\ w_1) ; (w_1 \supset \mathsf{fin}\ w_2) \equiv \Diamond(w_1 \wedge (w_1 \wedge \mathsf{fin}\ w_2))$	**FinChopEqvDiamond**
5	$\vdash$	$\Diamond(w_1 \wedge (w_1 \wedge \mathsf{fin}\ w_2)) \supset \mathsf{fin}\ w_2$	**PTL**
6	$\vdash$	$w \wedge (w \supset \mathsf{fin}\ w_1) ; (w_1 \supset \mathsf{fin}\ w_2) \supset \mathsf{fin}\ w_2$	1, 3, 4, 5, **Prop**
7	$\vdash$	$(w \supset \mathsf{fin}\ w_1) ; (w_1 \supset \mathsf{fin}\ w_2) \supset (w \supset \mathsf{fin}\ w_2)$	6, **Prop**

qed

$$\begin{array}{ll} & \vdash \quad w \wedge f \supset \mathsf{fin}\ w_1 \\ & \vdash \quad w_1 \wedge f_1 \supset \mathsf{fin}\ w_2 \\ \Rightarrow & \vdash \quad w \wedge (f ; f_1) \supset \mathsf{fin}\ w_2 \end{array}$$ ChopRule

Proof:

1	$\vdash$	$w \wedge (f\,;f_1) \supset (w \wedge f)\,;f_1$	**StateAndChopImport**
2	$\vdash$	$w \wedge f \ \supset \ \mathsf{fin}\ w_1$	given
3	$\vdash$	$(w \wedge f)\,;f_1 \supset (\mathsf{fin}\ w_1)\,;f_1$	2, **LeftChopImpChop**
4	$\vdash$	$(\mathsf{fin}\ w_1)\,;f_1 \equiv \Diamond(w_1 \wedge f_1)$	**FinChopEqvDiamond**
5	$\vdash$	$w_1 \wedge f_1 \supset \mathsf{fin}\ w_2$	given
6	$\vdash$	$\Diamond(w_1 \wedge f_1) \supset \Diamond\,\mathsf{fin}\ w_2$	5, **DiamondImpDiamond**
7	$\vdash$	$\Diamond\,\mathsf{fin}\ w_2 \supset \mathsf{fin}\ w_2$	**PTL**
8	$\vdash$	$w \wedge (f\,;f_1) \supset \mathsf{fin}\ w_2$	1, 3, 4, 6, 7, **Prop**

qed

$$\begin{array}{rl} & \vdash\ w \wedge f \ \supset\ f_1 \wedge \mathsf{fin}\ w_1 \\ & \vdash\ w_1 \wedge g \ \supset\ g_1 \\ \Rightarrow & \vdash\ w \wedge (f\,;g) \ \supset\ (f_1\,;g_1) \end{array}$$

ChopRep

Proof:

1	$\vdash$	$w \wedge f \supset f_1 \wedge \mathsf{fin}\ w_1$	given
2	$\vdash$	$w \wedge (f\,;g) \supset (f_1 \wedge \mathsf{fin}\ w)\,;g$	1, **StateAndChopImpChopRule**
3	$\vdash$	$(f_1 \wedge \mathsf{fin}\ w_1)\,;g \equiv f_1\,;(w_1 \wedge g)$	**AndFinChopEqvStateAndChop**
4	$\vdash$	$w_1 \wedge g \supset g_1$	given
5	$\vdash$	$f_1\,;(w_1 \wedge g) \supset f_1\,;g_1$	4, **RightChopImpChop**
6	$\vdash$	$w \wedge (f\,;g) \supset f_1\,;g_1$	2, 3, 5, **Prop**

qed

$$\begin{array}{rl} & \vdash\ w \wedge f \ \supset\ f_1 \wedge \mathsf{fin}\ w_1 \\ & \vdash\ w_1 \wedge g \ \supset\ g_1 \wedge \mathsf{fin}\ w_2 \\ \Rightarrow & \vdash\ w \wedge (f\,;g) \ \supset\ (f_1\,;g_1) \wedge \mathsf{fin}\ w_2 \end{array}$$

ChopRepAndFin

Proof:

1	$\vdash$	$w \wedge f \supset f_1 \wedge \mathsf{fin}\ w_1$	given
2	$\vdash$	$w_1 \wedge g \supset g_1 \wedge \mathsf{fin}\ w_2$	given
3	$\vdash$	$w \wedge (f\,;g) \supset f_1\,;(g_1 \wedge \mathsf{fin}\ w_2)$	1, 2, **ChopRep**
4	$\vdash$	$f_1\,;(g_1 \wedge \mathsf{fin}\ w_2) \supset f_1\,;g_1$	**ChopAndA**
5	$\vdash$	$f_1\,;(g_1 \wedge \mathsf{fin}\ w_2) \supset f_1\,;\mathsf{fin}\ w_2$	**ChopAndB**
6	$\vdash$	$f_1\,;\mathsf{fin}\ w_2 \supset \mathsf{fin}\ w_2$	**ChopFinImpFin**
7	$\vdash$	$w \wedge (f\,;g) \supset (f_1\,;g_1) \wedge \mathsf{fin}\ w_2$	3, 4, 5, 6, **Prop**

qed

The following lemma is used in **MoreChopLoop**.

$$\vdash \mathsf{true}\,;\mathsf{more} \equiv \mathsf{more}$$

TrueChopMoreEqvMore

$$\vdash f \ \supset\ \mathsf{more}\,;f \quad \Rightarrow \quad \neg f$$

MoreChopLoop

Proof:

1	$\vdash$	$f \supset \mathsf{more} ; f$	given
11	$\vdash$	$\Diamond f \supset \Diamond(\mathsf{more} ; f)$	**DiamondImpDiamond**
12	$\vdash$	$\Diamond(\mathsf{more} ; f) \equiv \mathsf{true} ; (\mathsf{more} ; f)$	def. of $\Diamond$
13	$\vdash$	$\mathsf{true} ; (\mathsf{more} ; f) \equiv (\mathsf{true} ; \mathsf{more}) ; f$	**ChopAssoc**
14	$\vdash$	$\Diamond(\mathsf{more} ; f) \equiv \mathsf{more} ; f$	**TrueChopMoreEqvMore**
2	$\vdash$	$\mathsf{more} ; f \equiv \bigcirc\Diamond f$	**MoreChopEqvNextDiamond**
3	$\vdash$	$\Diamond f \supset \bigcirc\Diamond f$	11, 14, 2, **Prop**
4	$\vdash$	$\neg(\Diamond f)$	3, **NextLoop**
5	$\vdash$	$\neg(\Diamond f) \supset \neg f$	**PTL**
6	$\vdash$	$\neg f$	4, 5, **MP**

qed

Here is a corollary:

$$\vdash f \wedge \neg g \ \supset \ (\mathsf{more} ; (f \wedge \neg g)) \quad \Rightarrow \quad \vdash f \supset g$$ MoreChopContra

Proof:

1	$\vdash$	$f \wedge \neg g \supset (\mathsf{more} ; (f \wedge \neg g))$	given
2	$\vdash$	$\neg(f \wedge \neg g)$	1, **MoreChopLoop**
3	$\vdash$	$f \supset g$	2, **Prop**

qed

Here is a variant of lemma **MoreChopLoop** that is useful in proofs:

$$\vdash f \ \supset \ g ; f, \quad \vdash g \ \supset \ \mathsf{more} \quad \Rightarrow \quad \neg f$$ ChopLoop

Proof:

1	$\vdash$	$f \supset g ; f$	given
2	$\vdash$	$g \supset \mathsf{more}$	given
3	$\vdash$	$g ; f \supset \mathsf{more} ; f$	2, **LeftChopImpChop**
4	$\vdash$	$f \supset \mathsf{more} ; f$	1, 3, **ImpChain**
5	$\vdash$	$\neg f$	4, **MoreChopLoop**

qed

Here is a variant of lemma **MoreChopContra** that is useful in proofs:

$$\vdash f \wedge \neg g \ \supset \ h ; f \wedge \neg(h ; g), \quad \vdash h \ \supset \ \mathsf{more} \quad \Rightarrow \quad \vdash f \supset g$$ ChopContra

Proof:

1	$\vdash$	$f \wedge \neg g \supset h \,;\, f \wedge \neg(h \,;\, g)$	given
2	$\vdash$	$h \supset \mathsf{more}$	given
3	$\vdash$	$h \,;\, f \wedge \neg(h \,;\, g) \supset h \,;\, (f \wedge \neg g)$	**ChopAndNotChopImp**
4	$\vdash$	$h \,;\, (f \wedge \neg g) \supset \mathsf{more} \,;\, (f \wedge \neg g)$	2, **LeftChopImpChop**
5	$\vdash$	$f \wedge \neg g \supset \mathsf{more} \,;\, (f \wedge \neg g)$	1, 3, 4, **ImpChain**
6	$\vdash$	$f \supset g$	5, **MoreChopContra**

qed

2.5 Properties of *chop-plus*

$\vdash \quad f \supset f^+$ ImpChopPlus

Proof:

1	$\vdash$	$f^+ \equiv f \vee (f \wedge \mathsf{more}) \,;\, f^+$	**ChopPlusEqv**
2	$\vdash$	$f \supset f^+$	1, **Prop**

qed

$\vdash \quad f \,;\, f^+ \supset f^+$ ChopChopPlusImpChopPlus

Proof:

1	$\vdash$	$\mathsf{empty} \vee \mathsf{more}$	**PTL**
2	$\vdash$	$f \supset \mathsf{empty} \vee (f \wedge \mathsf{more})$	1, **Prop**
3	$\vdash$	$f \,;\, f^+ \supset f^+ \vee (f \wedge \mathsf{more}) \,;\, f^+$	2, **EmptyOrChopImpRule**
4	$\vdash$	$f^+ \equiv f \vee (f \wedge \mathsf{more}) \,;\, f^+$	**ChopPlusEqv**
5	$\vdash$	$(f \wedge \mathsf{more}) \,;\, f^+ \supset f^+$	4, **Prop**
6	$\vdash$	$f \,;\, f^+ \supset f^+$	3, 5, **Prop**

qed

$\vdash \quad f^+ \equiv f \vee (f \,;\, f^+)$ ChopPlusEqvOrChopChopPlus

Proof for $\supset$:

1	$\vdash$	$f^+ \equiv f \vee (f \wedge \mathsf{more}) \,;\, f^+$	**ChopPlusEqv**
2	$\vdash$	$(f \wedge \mathsf{more}) \,;\, f^+ \supset f \,;\, f^+$	**AndChopA**
3	$\vdash$	$f^+ \supset f \vee f \,;\, f^+$	1, 2, **Prop**

qed

Proof for $\subset$:

1	$\vdash$	$f \supset f^+$	**ImpChopPlus**
2	$\vdash$	$\mathsf{empty} \lor \mathsf{more}$	**PTL**
3	$\vdash$	$f \supset \mathsf{empty} \lor (f \land \mathsf{more})$	2, **Prop**
4	$\vdash$	$f\, ;\, f^+ \supset f^+ \lor (f \land \mathsf{more})\, ;\, f^+$	3, **EmptyOrChopImpRule**
5	$\vdash$	$f^+ \equiv f \lor (f \land \mathsf{more})\, ;\, f^+$	**ChopPlusEqv**
6	$\vdash$	$(f \land \mathsf{more})\, ;\, f^+ \supset f^+$	5, **Prop**
7	$\vdash$	$f \lor (f\, ;\, f^+) \supset f^+$	1, 6, **Prop**

qed

$\vdash f \land \neg g \supset (g \land \mathsf{more})\, ;\, f \Rightarrow \ \vdash f \supset g^+$ ChopPlusIntro

Proof:

1	$\vdash$	$f \land \neg g \supset (g \land \mathsf{more})\, ;\, f$	given
2	$\vdash$	$g^+ \equiv g \lor (g \land \mathsf{more})\, ;\, g^+$	**ChopPlusEqv**
3	$\vdash$	$f \land \neg(g^+) \supset (g \land \mathsf{more})\, ;\, f \land \neg((g \land \mathsf{more})\, ;\, g^+)$	1, 2, **Prop**
4	$\vdash$	$g \land \mathsf{more} \supset \mathsf{more}$	**Prop**
5	$\vdash$	$f \supset g^+$	3, 4, **ChopContra**

qed

$\vdash f \ \supset\ g, \quad \vdash (f \land \mathsf{more})\, ;\, g \ \supset\ g \Rightarrow \ \vdash f^+ \ \supset\ g$ ChopPlusElim

Proof:

1	$\vdash$	$f^+ \equiv f \lor (f \land \mathsf{more})\, ;\, f^+$	**ChopPlusEqv**
2	$\vdash$	$f \ \supset\ g$	given
3	$\vdash$	$(f \land \mathsf{more})\, ;\, g \ \supset\ g$	given
4	$\vdash$	$f^+ \land \neg g \supset (f \land \mathsf{more})\, ;\, f^+ \land \neg((f \land \mathsf{more})\, ;\, g)$	1, 2, 3, **Prop**
5	$\vdash$	$f \land \mathsf{more} \supset \mathsf{more}$	**Prop**
6	$\vdash$	$f^+ \supset g$	4, 5, **ChopContra**

qed

$\vdash f \ \supset\ g, \quad \vdash f\, ;\, g \ \supset\ g \Rightarrow \ \vdash f^+ \ \supset\ g$ ChopPlusElimWithoutMore

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$f\, ;\, g \supset g$	given
3	$\vdash$	$(f \land \mathsf{more})\, ;\, g \supset f\, ;\, g$	**AndChopA**
4	$\vdash$	$(f \land \mathsf{more})\, ;\, g \supset g$	2, 3, **ImpChain**
5	$\vdash$	$f^+ \supset g$	1, 4, **ChopPlusElim**

qed

$\vdash f \ \supset\ g \Rightarrow \ \vdash f^+ \ \supset\ g^+$ ChopPlusImpChopPlus

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	**ChopPlusEqv**
3	$\vdash$	$g^+ \equiv g \vee (g \wedge \text{more}) ; g^+$	**ChopPlusEqv**
4	$\vdash$	$f^+ \wedge \neg(g^+) \supset ((f \wedge \text{more}) ; f^+) \wedge \neg((g \wedge \text{more}) ; g^+)$	1, 2, 3, **Prop**
5	$\vdash$	$f \wedge \text{more} \supset g \wedge \text{more}$	1, **Prop**
6	$\vdash$	$(f \wedge \text{more}) ; f^+ \supset (g \wedge \text{more}) ; f^+$	5, **LeftChopImpChop**
7	$\vdash$	$f^+ \wedge \neg(g^+) \supset ((g \wedge \text{more}) ; f^+) \wedge \neg((g \wedge \text{more}) ; g^+)$	4, 6, **Prop**
8	$\vdash$	$g \wedge \text{more} \supset \text{more}$	**Prop**
9	$\vdash$	$f^+ \supset g^+$	7, 8, **ChopContra**

qed

$\vdash f \equiv g \Rightarrow \ \vdash f^+ \equiv g^+$ ChopPlusEqvChopPlus

Proof:

1	$\vdash$	$f \equiv g$	given
2	$\vdash$	$f \supset g$	1, **Prop**
3	$\vdash$	$f^+ \supset g^+$	2, **ChopPlusImpChopPlus**
4	$\vdash$	$g \supset f$	1, **Prop**
5	$\vdash$	$g^+ \supset f^+$	4, **ChopPlusImpChopPlus**
6	$\vdash$	$f^+ \equiv g^+$	3, 5, **Prop**

qed

2.6 Properties of *chop-star*

$\vdash \quad f^* \equiv \text{empty} \vee (f \wedge \text{more}) ; f^*$ CSEqv

Proof:

1	$\vdash$	$f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	**ChopPlusEqv**
2	$\vdash$	$\text{empty} \equiv \neg\text{more}$	**PTL**
3	$\vdash$	$\text{empty} \vee f^+ \equiv \text{empty} \vee (f \wedge \text{more}) \vee (f \wedge \text{more}) ; f^+$	1, 2, **Prop**
4	$\vdash$	$(f \wedge \text{more}) ; \text{empty} \equiv f \wedge \text{more}$	**ChopEmpty**
5	$\vdash$	$(f \wedge \text{more}) ; (\text{empty} \vee f^+) \equiv$	
		$(f \wedge \text{more}) ; \text{empty} \vee (f \wedge \text{more}) ; f^+$	**ChopOrEqv**
6	$\vdash$	$\text{empty} \vee f^+ \equiv \text{empty} \vee ((f \wedge \text{more}) ; (\text{empty} \vee f^+))$	3, 4, 5, **Prop**
7	$\vdash$	$f^* \equiv \text{empty} \vee (f \wedge \text{more}) ; f^*$	6, def. of *

qed

$\vdash \quad \text{empty} \supset f^*$ EmptyImpCS

Proof:

1	$\vdash$	$f^* \equiv \text{empty} \lor f^+$	def. of *
2	$\vdash$	$\text{empty} \supset f^*$	1, **Prop**

qed

Here is an straightforward corollary:

$\vdash \quad \neg f^* \supset \text{more}$ NotCSImpMore

$\vdash \quad f^+ \supset f^*$ ChopPlusImpCS

Proof:

1	$\vdash$	$f^+ \supset \text{empty} \lor f^+$	**Prop**
2	$\vdash$	$f^+ \supset f^*$	1, def. of *

qed

$\vdash \quad f \supset f^*$ ImpCS

Proof:

1	$\vdash$	$f \supset f^+$	**ImpChopPlus**
2	$\vdash$	$f \supset \text{empty} \lor f^+$	1, **Prop**
3	$\vdash$	$f \supset f^*$	2, def. of *

qed

$\vdash \quad f^* ; g \equiv g \lor f^+ ; g$ CSChopEqvOrChopPlusChop

Proof:

1	$\vdash$	$f^* \equiv \text{empty} \lor f^+$	def. of *
2	$\vdash$	$f^* ; g \equiv g \lor f^+ ; g$	1, **EmptyOrChopEqvRule**

qed

$\vdash \quad f^* \equiv \text{empty} \lor (f ; f^*)$ CSEqvOrChopCS

Proof for $\supset$:

1	$\vdash$	$f^* \equiv \text{empty} \lor (f \land \text{more}) ; f^*$	**CSEqv**
2	$\vdash$	$(f \land \text{more}) ; f^* \supset f ; f^*$	**AndChopA**
3	$\vdash$	$f^* \supset \text{empty} \lor f ; f^*$	1, 2, **Prop**

qed

Proof for $\subset$:

1	$\vdash$	$\mathsf{empty} \supset f^*$	**EmptyImpCS**
2	$\vdash$	$\mathsf{empty} \lor \mathsf{more}$	**PTL**
3	$\vdash$	$f \supset \mathsf{empty} \lor (f \land \mathsf{more})$	2, **Prop**
4	$\vdash$	$f\,;f^* \supset f^* \lor (f \land \mathsf{more})\,;f^*$	3, **EmptyOrChopImpRule**
5	$\vdash$	$f^* \equiv f \lor (f \land \mathsf{more})\,;f^*$	**CSEqv**
6	$\vdash$	$(f \land \mathsf{more})\,;f^* \supset f^*$	5, **Prop**
7	$\vdash$	$\mathsf{empty} \lor (f\,;f^*) \supset f^*$	1, 6, **Prop**

qed

$\vdash \quad f\,;f^* \supset f^*$ ChopCSImpCS

Proof:

1	$\vdash$	$f^* \equiv \mathsf{empty} \lor (f\,;f^*)$	**CSEqvOrChopCS**
2	$\vdash$	$f\,;f^* \supset f^*$	1, **Prop**

qed

$\vdash \quad f^* \land \mathsf{more} \supset f^+$ CSAndMoreImpChopPlus

Proof:

1	$\vdash$	$f^* \land \mathsf{more} \supset f\,;f^*$	**CSAndMoreImpChopCS**
2	$\vdash$	$f^* \land \mathsf{more} \supset f^+$	1, def. of $^+$

qed

$\vdash \quad f^* \land \mathsf{more} \equiv (f \land \mathsf{more})\,;f^*$ CSAndMoreEqvAndMoreChop

Proof for $\supset$:

1	$\vdash$	$(\mathsf{empty} \lor (f \land \mathsf{more})\,;f^*) \land \mathsf{more} \supset (f \land \mathsf{more})\,;f^*$	**PTL**
2	$\vdash$	$f^* \land \mathsf{more} \supset (f \land \mathsf{more})\,;f^*$	1, def. of *

qed

Proof for $\subset$:

1	$\vdash$	$f^* \equiv \mathsf{empty} \lor (f \land \mathsf{more})\,;f^*$	**CSEqv**
2	$\vdash$	$(f \land \mathsf{more})\,;f^* \supset f^*$	1, **Prop**
3	$\vdash$	$(f \land \mathsf{more})\,;f^* \supset \mathsf{more}$	**MoreChopImpMore**
4	$\vdash$	$(f \land \mathsf{more})\,;f^* \supset f^* \land \mathsf{more}$	2, 3, **Prop**

qed

$\vdash \quad f^* \land \mathsf{more} \supset f\,;f^*$ CSAndMoreImpChopCS

Proof:

1	$\vdash$	$f^* \wedge \text{more} \equiv (f \wedge \text{more}) \,;\, f^*$	**CSAndMoreEqvAndMoreChop**
2	$\vdash$	$(f \wedge \text{more}) \,;\, f^* \supset f \,;\, f^*$	**AndChopA**
3	$\vdash$	$f^* \wedge \text{more} \supset f \,;\, f^*$	1, 2, **Prop**

qed

$\vdash \quad f^* \wedge \text{more} \supset f^* \,;\, f$ CSAndMoreImpCSChop

Proof:

1	$\vdash$	$f^* \wedge \text{more} \equiv (f \wedge \text{more}) \,;\, f^*$	**CSAndMoreEqvAndMoreChop**
2	$\vdash$	$\text{empty} \vee \text{more}$	**PTL**
3	$\vdash$	$f^* \supset \text{empty} \vee (f^* \wedge \text{more})$	2, **Prop**
4	$\vdash$	$(f \wedge \text{more}) \,;\, f^* \supset (f \wedge \text{more}) \vee ((f \wedge \text{more}) \,;\, (f^* \wedge \text{more}))$	3, **ChopEmptyOrImpRule**
5	$\vdash$	$f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}) \,;\, (f^* \wedge \text{more})$	**CSMoreNotImpChopCSAndMore**
6	$\vdash$	$f^* \equiv \text{empty} \vee (f \wedge \text{more}) \,;\, f^*$	**CSEqv**
7	$\vdash$	$f^* \,;\, f \equiv f \vee ((f \wedge \text{more}) \,;\, f^*) \,;\, f$	6, **EmptyOrChopEqvRule**
8	$\vdash$	$((f \wedge \text{more}) \,;\, f^*) \,;\, f \equiv (f \wedge \text{more}) \,;\, (f^* \,;\, f)$	**ChopAssoc**
9	$\vdash$	$(f^* \wedge \text{more}) \wedge \neg(f^* \,;\, f) \supset$ $(f \wedge \text{more}) \,;\, (f^* \wedge \text{more}) \wedge \neg((f \wedge \text{more}) \,;\, (f^* \,;\, f))$	5, 7, 8, **Prop**
10	$\vdash$	$f \wedge \text{more} \supset \text{more}$	**Prop**
11	$\vdash$	$f^* \wedge \text{more} \supset f^* \,;\, f$	9, 10, **ChopContra**

qed

The following lemma is used in the proof of **CSAndMoreImpCSChop**:

$\vdash \quad f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}) \,;\, (f^* \wedge \text{more})$ CSMoreNotImpChopCSAndMore

Proof:

1	$\vdash$	$f^* \wedge \text{more} \equiv (f \wedge \text{more}) \,;\, f^*$	**CSAndMoreEqvAndMoreChop**
2	$\vdash$	$\text{empty} \vee \text{more}$	**PTL**
3	$\vdash$	$f^* \supset \text{empty} \vee (f^* \wedge \text{more})$	2, **Prop**
4	$\vdash$	$(f \wedge \text{more}) \,;\, f^* \supset (f \wedge \text{more}) \vee ((f \wedge \text{more}) \,;\, (f^* \wedge \text{more}))$	3, **ChopEmptyOrImpRule**
5	$\vdash$	$(f \wedge \text{more}) \,;\, f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}) \,;\, (f^* \wedge \text{more})$	4, **Prop**
6	$\vdash$	$f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}) \,;\, (f^* \wedge \text{more})$	1, 5, **Prop**

qed

$\vdash \quad f^* \,;\, f^* \supset f^*$ CSChopCSImpCS

Proof:

1	$\vdash$	$f^* \equiv \mathsf{empty} \lor (f \land \mathsf{more}) ; f^*$	**CSEqv**
2	$\vdash$	$f^* ; f^* \equiv f^* \lor ((f \land \mathsf{more}) ; f^*) ; f^*$	1, **EmptyOrChopEqvRule**
3	$\vdash$	$f^* ; f^* \land \neg(f^*) \supset ((f \land \mathsf{more}) ; f^*) ; f^* \land \neg((f \land \mathsf{more}) ; f^*)$	1, 2, **Prop**
4	$\vdash$	$((f \land \mathsf{more}) ; f^*) ; f^* \equiv (f \land \mathsf{more}) ; f^* ; f^*$	**ChopAssoc**
5	$\vdash$	$f^* ; f^* \land \neg(f^*) \supset (f \land \mathsf{more}) ; f^* ; f^* \land \neg((f \land \mathsf{more}) ; f^*)$	3, 4, **Prop**
6	$\vdash$	$f \land \mathsf{more} \supset \mathsf{more}$	**Prop**
7	$\vdash$	$f^* ; f^* \supset f^*$	5, 6, **ChopContra**

qed

$\vdash \quad f^* ; f \supset f^*$ CSChopImpCS

Proof:

1	$\vdash$	$f \supset f^*$	**ImpCS**
2	$\vdash$	$f^* ; f \supset f^* ; f^*$	1, **LeftChopImpChop**
3	$\vdash$	$f^* ; f^* \supset f^*$	**CSChopCSImpCS**
4	$\vdash$	$f^* ; f \supset f^*$	2, 3, **ImpChain**

qed

$\vdash \quad (f^*)^* \supset f^*$ CSCSImpCS

Proof:

1	$\vdash$	$\mathsf{empty} \supset f^*$	**EmptyImpCS**
2	$\vdash$	$(f^* \land \mathsf{more}) ; f^* \supset f^* ; f^*$	**AndChopA**
3	$\vdash$	$f^* ; f^* \supset f^*$	**CSChopCSImpCS**
4	$\vdash$	$(f^* \land \mathsf{more}) ; f^* \supset f^*$	2, 3, **ImpChain**
5	$\vdash$	$(f^*)^* \supset f^*$	1, 4, **CSElim**

qed

$\vdash f \ \supset \ g \Rightarrow \quad \vdash f^* \ \supset \ g^*$ CSImpCS

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$f^+ \supset g^+$	1, **ChopPlusImpChopPlus**
3	$\vdash$	$\mathsf{empty} \lor f^+ \supset \mathsf{empty} \lor g^+$	2, **Prop**
4	$\vdash$	$f^* \supset g^*$	3, def. of *

qed

$\vdash f \equiv g \Rightarrow \quad \vdash f^* \equiv g^*$ CSEqvCS

Proof:

1	$\vdash$	$f \equiv g$	given
2	$\vdash$	$f^+ \equiv g^+$	1, **ChopPlusEqvChopPlus**
3	$\vdash$	$\mathsf{empty} \lor f^+ \equiv \mathsf{empty} \lor g^+$	2, **Prop**
4	$\vdash$	$f^* \equiv g^*$	3, def. of *

qed

$$\vdash \quad (f \land g)^* \supset f^* \qquad \text{AndCSA}$$

Proof:

1	$\vdash$	$f \land g \supset f$	**Prop**
2	$\vdash$	$(f \land g)^* \supset f^*$	1, **CSImpCS**

qed

$$\vdash \quad (f \land g)^* \supset g^* \qquad \text{AndCSB}$$

Proof:

1	$\vdash$	$f \land g \supset g$	**Prop**
2	$\vdash$	$(f \land g)^* \supset g^*$	1, **CSImpCS**

qed

$$\vdash f \land \mathsf{more} \ \supset \ (g \land \mathsf{more})\,;f \Rightarrow \ \vdash f \ \supset \ g^* \qquad \text{CSIntro}$$

Proof:

1	$\vdash$	$f \land \mathsf{more} \supset (g \land \mathsf{more})\,;f$	given
2	$\vdash$	$\mathsf{more} \equiv \neg\mathsf{empty}$	**PTL**
3	$\vdash$	$f \land \neg\mathsf{empty} \supset (g \land \mathsf{more})\,;f$	1, 2, **Prop**
4	$\vdash$	$g^* \equiv \mathsf{empty} \lor (g \land \mathsf{more})\,;g^*$	**CSEqv**
5	$\vdash$	$f \land \neg g^* \supset (g \land \mathsf{more})\,;f \land \neg((g \land \mathsf{more})\,;g^*)$	3, 4, **Prop**
6	$\vdash$	$g \land \mathsf{more} \supset \mathsf{more}$	**Prop**
7	$\vdash$	$f \supset g^*$	5, 6, **ChopContra**

qed

$$\vdash \mathsf{empty} \ \supset \ g, \quad \vdash (f \land \mathsf{more})\,;g \ \supset \ g \Rightarrow \ \vdash f^* \ \supset \ g \qquad \text{CSElim}$$

Proof:

1	$\vdash$	$f^* \equiv \mathsf{empty} \lor (f \land \mathsf{more})\,;f^*$	**CSEqv**
2	$\vdash$	$\mathsf{empty} \ \supset \ g$	given
3	$\vdash$	$(f \land \mathsf{more})\,;g \ \supset \ g$	given
4	$\vdash$	$f^* \land \neg g \supset (f \land \mathsf{more})\,;f^* \land \neg((f \land \mathsf{more})\,;g)$	1, 2, 3, **Prop**
5	$\vdash$	$f \land \mathsf{more} \supset \mathsf{more}$	**Prop**
6	$\vdash$	$f^* \supset g$	4, 5, **ChopContra**

qed

$$\vdash \text{empty} \supset g, \quad \vdash f \,;\, g \supset g \Rightarrow \quad \vdash f^* \supset g$$ CSElimWithoutMore

Proof:

1	$\vdash$	$\text{empty} \supset g$	given
2	$\vdash$	$f \,;\, g \supset g$	given
3	$\vdash$	$(f \wedge \text{more}) \,;\, g \supset f \,;\, g$	AndChopA
4	$\vdash$	$(f \wedge \text{more}) \,;\, g \supset g$	2, 3, ImpChain
5	$\vdash$	$f^* \supset g$	1, 4, CSElim

qed

$$\vdash f \equiv g^* \,;\, h \Rightarrow \quad \vdash f \equiv (g \,;\, f) \vee h$$ CSChopEqvChopOrRule

Proof:

1	$\vdash$	$f \equiv g^* \,;\, h$	given
2	$\vdash$	$g^* \equiv \text{empty} \vee (g \,;\, g^*)$	CSEqvOrChopCS
3	$\vdash$	$g^* \,;\, h \equiv h \vee ((g \,;\, g^*) \,;\, h)$	2, EmptyOrChopEqvRule
4	$\vdash$	$(g \,;\, g^*) \,;\, h \equiv g \,;\, (g^* \,;\, h)$	*ChopAssoc*
5	$\vdash$	$g \,;\, f \equiv g \,;\, (g^* \,;\, h)$	1, RightChopEqvChop
6	$\vdash$	$f \equiv (g \,;\, f) \vee h$	1, 3, 4, 5, Prop

qed

$$\vdash f \wedge \neg h \supset g \,;\, f, \quad \vdash g \supset \text{more} \Rightarrow \quad f \supset g^* \,;\, h$$ CSChopIntroRule

Proof:

1	$\vdash$	$f \wedge \neg h \supset g \,;\, f$	given
2	$\vdash$	$g \supset \text{more}$	given
3	$\vdash$	$g \supset g \wedge \text{more}$	2, Prop
4	$\vdash$	$g \,;\, f \supset (g \wedge \text{more}) \,;\, f$	3, LeftChopImpChop
5	$\vdash$	$f \supset (g \wedge \text{more}) \,;\, f \vee h$	1, 4, Prop
6	$\vdash$	$g^* \equiv \text{empty} \vee (g \wedge \text{more}) \,;\, g^*$	CSEqv
7	$\vdash$	$g^* \,;\, h \equiv h \vee ((g \wedge \text{more}) \,;\, g^*) \,;\, h$	6, EmptyOrChopImpRule
8	$\vdash$	$((g \wedge \text{more}) \,;\, g^*) \,;\, h \equiv (g \wedge \text{more}) \,;\, (g^* \,;\, h)$	ChopAssoc
9	$\vdash$	$g^* \,;\, h \equiv h \vee (g \wedge \text{more}) \,;\, (g^* \,;\, h)$	7, 8, Prop
10	$\vdash$	$f \wedge \neg(g^* \,;\, h) \supset (g \wedge \text{more}) \,;\, f \wedge \neg((g \wedge \text{more}) \,;\, (g^* \,;\, h))$	5, 9, Prop
11	$\vdash$	$g \wedge \text{more} \supset \text{more}$	Prop
12	$\vdash$	$f \supset g^* \,;\, h$	10, 11, ChopContra

qed

$$\vdash f \supset \text{empty} \vee (\Box\, w \wedge \text{more}) \,;\, f \Rightarrow \quad \vdash w \wedge f \supset \Box\, w$$ CSImpBox

Proof:

1	$\vdash$	$f \supset \mathrm{empty} \lor (\Box\, w \land \mathrm{more})\,;\, f$	given
2	$\vdash$	$w \land \neg \Box\, w \supset \neg\mathrm{empty}$	**PTL**
3	$\vdash$	$w \land f \land \neg \Box\, w \supset (\Box\, w \land \mathrm{more})\,;\, f$	1, 2, **Prop**
4	$\vdash$	$\Box\, w \land \mathrm{more} \supset (\Box\, w \land \mathrm{more}) \land \mathrm{fin}\ w$	**PTL**
5	$\vdash$	$(\Box\, w \land \mathrm{more})\,;\, f \supset ((\Box\, w \land \mathrm{more}) \land \mathrm{fin}\ w)\,;\, f$	4, **LeftChopImpChop**
6	$\vdash$	$((\Box\, w \land \mathrm{more}) \land \mathrm{fin}\ w)\,;\, f \equiv (\Box\, w \land \mathrm{more})\,;\, (w \land f)$	**AndFinChopEqvStateAndChop**
7	$\vdash$	$\neg \Box\, w \supset (\Box\, w) \rightsquigarrow \neg \Box\, w$	**NotBoxStateImpBoxYieldsNotBo**
8	$\vdash$	$(\Box\, w) \rightsquigarrow \neg \Box\, w \supset (\Box\, w \land \mathrm{more}) \rightsquigarrow \neg \Box\, w$	**AndYieldsA**
9	$\vdash$	$(\Box\, w \land \mathrm{more})\,;\, (w \land f) \land (\Box\, w \land \mathrm{more}) \rightsquigarrow \neg \Box\, w \supset$ $(\Box\, w \land \mathrm{more})\,;\, ((w \land f) \land \neg \Box\, w)$	**ChopAndYieldsImp**
10	$\vdash$	$(w \land f) \land \neg \Box\, w \supset (\Box\, w \land \mathrm{more})\,;\, ((w \land f) \land \neg \Box\, w)$	3, 5, 6, 7, 8, 9, **Prop**
11	$\vdash$	$(\Box\, w \land \mathrm{more})\,;\, ((w \land f) \land \neg \Box\, w) \supset \mathrm{more}\,;\, ((w \land f) \land \neg \Box\, w)$	**AndChopB**
12	$\vdash$	$(w \land f) \land \neg \Box\, w \supset \mathrm{more}\,;\, ((w \land f) \land \neg \Box\, w)$	10, 11, **ImpChain**
13	$\vdash$	$w \land f \supset \Box\, w$	**MoreChopContra**

qed

> $\vdash \quad w \land (\Box\, w)^* \equiv \Box\, w$ — BoxCSEqvBox

Proof for $\supset$:

1	$\vdash$	$(\Box\, w)^* \equiv \mathrm{empty} \lor (\Box\, w \land \mathrm{more})\,;\, (\Box\, w)^*$	**CSEqv**
2	$\vdash$	$(\Box\, w)^* \supset \mathrm{empty} \lor (\Box\, w \land \mathrm{more})\,;\, (\Box\, w)^*$	1, **Prop**
3	$\vdash$	$w \land (\Box\, w)^* \supset \Box\, w$	2, **CSImpBox**

qed

Proof for $\subset$:

1	$\vdash$	$\Box\, w \supset w$	**PTL**
2	$\vdash$	$\Box\, w \supset (\Box\, w)^*$	**ImpCS**
3	$\vdash$	$\Box\, w \supset w \land (\Box\, w)^*$	1, 2, **Prop**

qed

> $\vdash \Box\, w \land f^* \equiv w \land (\Box\, w \land f)^*$ — BoxStateAndCSEqvCS

Proof for $\supset$:

1	$\vdash$	$\Box\, w \supset w$	**PTL**
2	$\vdash$	$f^* \land \mathrm{more} \equiv (f \land \mathrm{more})\,;\, f^*$	**CSAndMoreEqvAndMoreChop**
3	$\vdash$	$\Box\, w \land ((f \land \mathrm{more})\,;\, f^*) \equiv (\Box\, w \land f \land \mathrm{more})\,;\, (\Box\, w \land f^*)$	**BoxStateAndChopEqvChop**
4	$\vdash$	$\Box\, w \land f \land \mathrm{more} \supset (\Box\, w \land f) \land \mathrm{more}$	**PTL**
5	$\vdash$	$(\Box\, w \land f \land \mathrm{more})\,;\, (\Box\, w \land f^*) \supset$ $((\Box\, w \land f) \land \mathrm{more})\,;\, (\Box\, w \land f^*)$	4, **LeftChopImpChop**
6	$\vdash$	$(\Box\, w \land f^*) \land \mathrm{more} \supset ((\Box\, w \land f) \land \mathrm{more})\,;\, (\Box\, w \land f^*)$	2, 3, 5, **Prop**
7	$\vdash$	$\Box\, w \land f^* \supset (\Box\, w \land f)^*$	6, **CSIntro**
8	$\vdash$	$\Box\, w \land f^* \supset w \land (\Box\, w \land f)^*$	1, 7, **Prop**

qed

Proof for $\subset$:

1	$\vdash$	$(\Box\, w \wedge f)^* \supset (\Box\, w)^*$	**AndCSA**
2	$\vdash$	$w \wedge (\Box\, w)^* \equiv \Box\, w$	**BoxCSEqvBox**
3	$\vdash$	$(\Box\, w \wedge f)^* \supset f^*$	**AndCSB**
4	$\vdash$	$w \wedge (\Box\, w \wedge f)^* \supset \Box\, w \wedge f^*$	1, 2, 3, **Prop**

qed

See also the lemma **BoxStateAndChopEqvChop** for *chop*.

$$\vdash \quad \boxdot(f \supset g) \supset f^* \supset g^* \qquad \text{BaCSImpCS}$$

Proof:

1	$\vdash$	$f^* \equiv \mathsf{empty} \vee (f \wedge \mathsf{more})\,;f^*$	**CSEqv**
2	$\vdash$	$g^* \equiv \mathsf{empty} \vee (g \wedge \mathsf{more})\,;g^*$	**CSEqv**
3	$\vdash$	$f^* \wedge \neg(g^*) \supset (f \wedge \mathsf{more})\,;f^* \wedge \neg((g \wedge \mathsf{more})\,;g^*)$	1, 2, **Prop**
4	$\vdash$	$(f \supset g) \supset (f \wedge \mathsf{more} \supset g \wedge \mathsf{more})$	**Prop**
5	$\vdash$	$\boxdot(f \supset g) \supset \boxdot(f \wedge \mathsf{more} \supset g \wedge \mathsf{more})$	4, **BaImpBa**
6	$\vdash$	$\boxdot(f \wedge \mathsf{more} \supset g \wedge \mathsf{more}) \supset (f \wedge \mathsf{more})\,;f^* \supset (g \wedge \mathsf{more})\,;f^*$	**BaLeftChopImpChop**
7	$\vdash$	$\boxdot(f \supset g) \wedge (f \wedge \mathsf{more})\,;f^* \supset (g \wedge \mathsf{more})\,;f^*$	5, 6, **Prop**
8	$\vdash$	$(g \wedge \mathsf{more})\,;f^* \wedge \neg((g \wedge \mathsf{more})\,;g^*) \supset (g \wedge \mathsf{more})\,;(f^* \wedge \neg(g^*))$	**ChopAndNotChopImp**
9	$\vdash$	$(g \wedge \mathsf{more})\,;(f^* \wedge \neg(g^*)) \supset \mathsf{more}\,;(f^* \wedge \neg(g^*))$	**AndChopB**
10	$\vdash$	$\boxdot(f \supset g) \supset \mathsf{more}\,;(f^* \wedge \neg(g^*)) \supset \mathsf{more}\,;(\boxdot(f \supset g) \wedge f^* \wedge \neg(g^*))$	**BaChopImpChopBa**
11	$\vdash$	$\boxdot(f \supset g) \wedge f^* \wedge \neg(g^*) \supset \mathsf{more}\,;(\boxdot(f \supset g) \wedge f^* \wedge \neg(g^*))$	3, 7, 8, 9, 10, **Prop**
12	$\vdash$	$\neg(\boxdot(f \supset g) \wedge f^* \wedge \neg(g^*))$	11, **MoreChopLoop**
13	$\vdash$	$\boxdot(f \supset g) \supset f^* \supset g^*$	12, **Prop**

qed

The following corollary can be readily verified:

$$\vdash \quad \boxdot(f \equiv g) \supset f^* \equiv g^* \qquad \text{BaCSEqvCS}$$

$$\vdash \quad \boxdot\, f \wedge g^* \supset (f \wedge g)^* \qquad \text{BaAndCSImport}$$

Proof:

1	$\vdash$	$f \supset (g \supset f \wedge g)$	**Prop**
2	$\vdash$	$\boxdot\, f \supset \boxdot(g \supset f \wedge g)$	1, **BaImpBa**
3	$\vdash$	$\boxdot(g \supset f \wedge g) \supset g^* \supset (f \wedge g)^*$	**BaCSImpCS**
4	$\vdash$	$\boxdot\, f \wedge g^* \supset (f \wedge g)^*$	2, 3, **Prop**

qed

2.7 Properties of While

$\vdash$ while w do $f \equiv$ if w then f ; (while w do f) else empty **WhileEqvIf**

Proof:

1	$\vdash$	while w do $f \equiv (w \wedge f)^* \wedge$ fin $\neg w$	def. of while
2	$\vdash$	$(w \wedge f)^* \equiv$ empty $\vee ((w \wedge f) ; (w \wedge f)^*)$	CSEqvOrChopCS
3	$\vdash$	empty $\wedge$ fin $\neg w \equiv \neg w \wedge$ empty	PTL
4	$\vdash$	$(w \wedge f) ; (w \wedge f)^* \equiv w \wedge f ; (w \wedge f)^*$	StateAndChop
5	$\vdash$	$(f ; (w \wedge f)^*) \wedge$ fin $\neg w \equiv f ; ((w \wedge f)^* \wedge$ fin $\neg w)$	ChopAndFin
6	$\vdash$	$f ; ((w \wedge f)^* \wedge$ fin $\neg w) \equiv f$; while w do f	def. of while
7	$\vdash$	while w do f $\equiv (\neg w \wedge$ empty$) \vee (w \wedge (f$; while w do $f))$	1, 2, 3, 4, 5, 6, Prop
8	$\vdash$	while w do $f \equiv$ if w then f ; (while w do f) else empty	7, Prop

qed

$\vdash$ (while w do f) ; $g \equiv$ if w then f ; (while w do f) ; g else g **WhileChopEqvIf**

Proof:

1	$\vdash$	while w do $f \equiv$ if w then f ; (while w do f) else empty	WhileEqvIf
2	$\vdash$	(while w do f) ; $g \equiv$ if w then (f ; while w do f) ; g else empty ; g	1, IfChopEqvRule
3	$\vdash$	(f ; while w do f) ; $g \equiv f$; (while w do f) ; g	ChopAssoc
4	$\vdash$	empty ; $g \equiv g$	EmptyChop
5	$\vdash$	(while w do f) ; $g \equiv$ if w then f ; (while w do f) ; g else g	2, 3, 4, Prop

qed

$\vdash f \equiv$ (while w do g) ; $h \Rightarrow \quad f \equiv$ if w then g ; f else h **WhileChopEqvIfRule**

Proof:

1	$\vdash$	$f \equiv$ (while w do g) ; h	given
2	$\vdash$	(while w do g) ; $h \equiv$ if w then g ; (while w do g) ; h else h	WhileChopEqvIf
3	$\vdash$	g ; $f \equiv g$; (while w do g) ; h	1, RightChopEqvChop
4	$\vdash$	$f \equiv$ if w then g ; f else h	1, 2, 3, Prop

qed

$\vdash$ while w do $f \supset$ fin $\neg w$ **WhileImpFin**

Proof:

1	$\vdash$	$(w \wedge f)^* \wedge \text{fin } \neg w \supset \text{fin } \neg w$	**Prop**
2	$\vdash$	$\text{while } w \text{ do } f \supset \text{fin } \neg w$	1, def. of while

qed

> $\vdash \quad \text{while } w \text{ do } f \equiv (\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}) ; \text{while } w \text{ do } f)$
>
> WhileEqvEmptyOrChopWhile

Proof:

1	$\vdash$	$(w \wedge f)^* \equiv \text{empty} \vee ((w \wedge f) \wedge \text{more}) ; (w \wedge f)^*$	**CSEqv**
2	$\vdash$	$(w \wedge f) \wedge \text{more} \equiv w \wedge (f \wedge \text{more})$	**Prop**
3	$\vdash$	$((w \wedge f) \wedge \text{more}) ; (w \wedge f)^* \equiv (w \wedge f \wedge \text{more}) ; (w \wedge f)^*$	2, **LeftChopEqvChop**
4	$\vdash$	$(w \wedge f)^* \equiv \text{empty} \vee (w \wedge f \wedge \text{more}) ; (w \wedge f)^*$	1, 3, **Prop**
5	$\vdash$	$(w \wedge f)^* \wedge \text{fin } \neg w \equiv$	
		$(\text{empty} \wedge \text{fin } \neg w) \vee ((w \wedge f \wedge \text{more}) ; (w \wedge f)^* \wedge \text{fin } \neg w)$	1, **Prop**
6	$\vdash$	$\text{empty} \wedge \text{fin } \neg w \equiv \neg w \wedge \text{empty}$	**PTL**
7	$\vdash$	$(w \wedge f \wedge \text{more}) ; (w \wedge f)^* \equiv w \wedge (f \wedge \text{more}) ; (w \wedge f)^*$	**StateAndChop**
8	$\vdash$	$(f \wedge \text{more}) ; (w \wedge f)^* \wedge \text{fin } \neg w \equiv$	
		$(f \wedge \text{more}) ; ((w \wedge f)^* \wedge \text{fin } \neg w)$	**ChopAndFin**
9	$\vdash$	$(w \wedge f)^* \wedge \text{fin } \neg w \equiv$	
		$(\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}) ; ((w \wedge f)^* \wedge \text{fin } \neg w))$	5, 6, 7, 8, **Prop**
10	$\vdash$	$\text{while } w \text{ do } f \equiv$	
		$(\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}) ; \text{while } w \text{ do } f)$	9, def. of while

qed

> $\vdash \quad \neg w \wedge f \supset \text{empty}$ WhileIntro
>
> $\vdash \quad w \wedge f \supset (g \wedge \text{more}) ; f$
>
> $\Rightarrow \quad \vdash \quad f \supset \text{while } w \text{ do } g$

Proof:

1	$\vdash$	$\neg w \wedge f \supset \text{empty}$	given
2	$\vdash$	$w \wedge f \supset (g \wedge \text{more}) ; f$	given
3	$\vdash$	$\text{while } w \text{ do } g \equiv$	
		$(\neg w \wedge \text{empty}) \vee (w \wedge (g \wedge \text{more}) ; \text{while } w \text{ do } g)$	**WhileEqvEmptyOrChopWhile**
4	$\vdash$	$f \wedge \neg \text{while } w \text{ do } g \supset$	
		$(g \wedge \text{more}) ; f \wedge \neg((g \wedge \text{more}) ; \text{while } w \text{ do } g)$	1, 2, 3, **Prop**
5	$\vdash$	$g \wedge \text{more} \supset \text{more}$	**Prop**
6	$\vdash$	$f \supset \text{while } w \text{ do } g$	4, 5, **ChopContra**

qed

$\vdash \neg w \wedge \text{empty} \supset g$ WhileElim
$\vdash w \wedge (f \wedge \text{more}) ; g \supset g$
$\Rightarrow \vdash \text{while } w \text{ do } f \supset g$

Proof:

1	$\vdash$	$\text{while } w \text{ do } f \equiv$ $(\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}) ; \text{while } w \text{ do } f)$	**WhileEqvEmptyOrChopWhile**
2	$\vdash$	$\neg w \wedge \text{empty} \supset g$	given
3	$\vdash$	$w \wedge (f \wedge \text{more}) ; g \supset g$	given
4	$\vdash$	$\text{while } w \text{ do } f \wedge \neg g \supset$ $(f \wedge \text{more}) ; \text{while } w \text{ do } f \wedge \neg((f \wedge \text{more}) ; g)$	1, 2, 3, **Prop**
5	$\vdash$	$f \wedge \text{more} \supset \text{more}$	**Prop**
6	$\vdash$	$\text{while } w \text{ do } f \supset g$	4, 5, **ChopContra**

qed

$\vdash ⓐ(f \supset g) \supset (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$ BaWhileImpWhile

Proof:

1	$\vdash$	$(f \supset g) \supset ((w \wedge f) \supset (w \wedge g)$	**Prop**
2	$\vdash$	$ⓐ(f \supset g) \supset ⓐ((w \wedge f) \supset (w \wedge g))$	1, **BaImpBa**
3	$\vdash$	$ⓐ((w \wedge f) \supset (w \wedge g)) \supset ((w \wedge f)^* \supset (w \wedge g)^*)$	**BaCSImpCS**
4	$\vdash$	$ⓐ(f \supset g) \supset ((w \wedge f)^* \wedge \text{fin } \neg w \supset (w \wedge g)^* \wedge \text{fin } \neg w)$	2, 3, **Prop**
5	$\vdash$	$ⓐ(f \supset g) \supset (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$	4, def. of while

qed

$\vdash f \supset g \Rightarrow \vdash (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$ WhileImpWhile

Proof:

1	$\vdash$	$f \supset g$	given
2	$\vdash$	$ⓐ(f \supset g)$	1, **BaGen**
3	$\vdash$	$ⓐ(f \supset g) \supset (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$	**BaWhileImpWhile**
4	$\vdash$	$(\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$	2, 3, **MP**

qed

2.8 Properties of Halt

$\vdash \text{halt } w ; f \equiv \text{if } w \text{ then } f \text{ else } \bigcirc(\text{halt } w ; f)$ HaltChopEqv

Proof:

1	$\vdash$	$\text{halt } w \equiv \text{if } w \text{ then empty else } \bigcirc \text{halt } w$	**PTL**
2	$\vdash$	$\text{halt } w \,;\, f \equiv \text{if } w \text{ then empty} \,;\, f \text{ else } (\bigcirc \text{halt } w) \,;\, f$	1, **IfChopEqvRule**
3	$\vdash$	$\text{empty} \,;\, f \equiv f$	**EmptyChop**
4	$\vdash$	$(\bigcirc \text{halt } w) \,;\, f \equiv \bigcirc(\text{halt } w \,;\, f)$	**NextChop**
5	$\vdash$	$\text{halt } w \,;\, f \equiv \text{if } w \text{ then } f \text{ else } \bigcirc(\text{halt } w \,;\, f)$	2, 3, 4, **Prop**

qed

$\vdash \quad w \wedge (\text{halt } w \,;\, f) \supset f$ AndHaltChopImp

Proof:

1	$\vdash$	$\text{halt } w \,;\, f \equiv \text{if } w \text{ then } f \text{ else } \bigcirc(\text{halt } w \,;\, f)$	**HaltChopEqv**
2	$\vdash$	$w \wedge (\text{halt } w \,;\, f) \supset f$	1, **Prop**

qed

$\vdash \quad \neg w \wedge (\text{halt } w \,;\, f) \supset \bigcirc(\text{halt } w \,;\, f)$ NotAndHaltChopImpNext

Proof:

1	$\vdash$	$\text{halt } w \,;\, f \equiv \text{if } w \text{ then } f \text{ else } \bigcirc(\text{halt } w \,;\, f)$	**HaltChopEqv**
2	$\vdash$	$\neg w \wedge (\text{halt } w \,;\, f) \supset \bigcirc(\text{halt } w \,;\, f)$	1, **Prop**

qed

$\vdash \quad \neg w \wedge (\text{halt } w \,;\, f) \supset \text{skip} \leadsto\!\!> (\text{halt } w \,;\, f)$ NotAndHaltChopImpSkipYields

Proof:

1	$\vdash$	$\neg w \wedge (\text{halt } w \,;\, f) \supset \bigcirc(\text{halt } w \,;\, f)$	**NotAndHaltChopImpNext**
2	$\vdash$	$\bigcirc(\text{halt } w \,;\, f) \supset \text{skip} \leadsto\!\!> (\text{halt } w \,;\, f)$	**NextImpSkipYields**
3	$\vdash$	$\neg w \wedge (\text{halt } w \,;\, f) \supset \text{skip} \leadsto\!\!> (\text{halt } w \,;\, f)$	1, 2, **ImpChain**

qed

$\vdash \quad \text{halt } w \,;\, f \supset \neg(\text{halt } w \,;\, \neg f)$ HaltChopImpNotHaltChopNot

Proof:

1	$\vdash$	$\mathsf{halt}\ w ; f \equiv \mathsf{if}\ w\ \mathsf{then}\ f\ \mathsf{else}\ \bigcirc(\mathsf{halt}\ w ; f)$	HaltChopEqv
2	$\vdash$	$\mathsf{if}\ w\ \mathsf{then}\ f\ \mathsf{else}\ \bigcirc(\mathsf{halt}\ w ; f) \supset ((w \supset f) \wedge (\neg w \supset \bigcirc(\mathsf{halt}\ w ; f)))$	Prop
3	$\vdash$	$\mathsf{halt}\ w ; \neg f \equiv \mathsf{if}\ w\ \mathsf{then}\ \neg f\ \mathsf{else}\ \bigcirc(\mathsf{halt}\ w ; \neg f)$	HaltChopEqv
4	$\vdash$	$\mathsf{if}\ w\ \mathsf{then}\ \neg f\ \mathsf{else}\ \bigcirc(\mathsf{halt}\ w ; \neg f) \supset ((w \supset \neg f) \wedge (\neg w \supset \bigcirc(\mathsf{halt}\ w ; \neg f)))$	Prop
5	$\vdash$	$(\mathsf{halt}\ w ; f) \wedge (\mathsf{halt}\ w ; \neg f) \supset$ $(w \supset f) \wedge (\neg w \supset \bigcirc(\mathsf{halt}\ w ; f)) \wedge (w \supset \neg f) \wedge (\neg w \supset \bigcirc(\mathsf{halt}\ w ; \neg f))$	1, 2, 3, 4, Prop
6	$\vdash$	$(\mathsf{halt}\ w ; f) \wedge (\mathsf{halt}\ w ; \neg f) \supset \bigcirc(\mathsf{halt}\ w ; f) \wedge \bigcirc(\mathsf{halt}\ w ; \neg f)$	6, Prop
7	$\vdash$	$\bigcirc(\mathsf{halt}\ w ; f) \wedge \bigcirc(\mathsf{halt}\ w ; \neg f) \equiv \bigcirc((\mathsf{halt}\ w ; f) \wedge (\mathsf{halt}\ w ; \neg f))$	PTL
8	$\vdash$	$(\mathsf{halt}\ w ; f) \wedge (\mathsf{halt}\ w ; \neg f) \supset \bigcirc((\mathsf{halt}\ w ; f) \wedge (\mathsf{halt}\ w ; \neg f))$	6, 7, Prop
9	$\vdash$	$\neg((\mathsf{halt}\ w ; f) \wedge (\mathsf{halt}\ w ; \neg f))$	8.NextLoop
10	$\vdash$	$\mathsf{halt}\ w ; f \supset \neg(\mathsf{halt}\ w ; \neg f)$	9, Prop

qed

$\vdash \quad \mathsf{halt}\ w ; f \supset (\mathsf{halt}\ w) \rightsquigarrow f$ HaltChopImpHaltYields

Proof:

1	$\vdash$	$\mathsf{halt}\ w ; f \supset \neg(\mathsf{halt}\ w ; \neg f)$	HaltChopImpNotHaltChopNot
2	$\vdash$	$\mathsf{halt}\ w ; f \supset (\mathsf{halt}\ w) \rightsquigarrow f$	1, def. of $\rightsquigarrow$

qed

$\vdash \quad (\mathsf{halt}\ w) ; f \wedge (\mathsf{halt}\ w) ; g \supset (\mathsf{halt}\ w) ; (f \wedge g)$ HaltChopAnd

Proof:

1	$\vdash$	$(\mathsf{halt}\ w) ; g \supset (\mathsf{halt}\ w) \rightsquigarrow g$	HaltChopImpHaltYields
2	$\vdash$	$(\mathsf{halt}\ w) ; f \wedge (\mathsf{halt}\ w) ; g \supset (\mathsf{halt}\ w) ; f \wedge (\mathsf{halt}\ w) \rightsquigarrow g$	1, 2, Prop
3	$\vdash$	$(\mathsf{halt}\ w) ; f \wedge (\mathsf{halt}\ w) \rightsquigarrow g \supset (\mathsf{halt}\ w) ; (f \wedge g)$	ChopAndYieldsImp
4	$\vdash$	$(\mathsf{halt}\ w) ; f \wedge (\mathsf{halt}\ w) ; g \supset (\mathsf{halt}\ w) ; (f \wedge g)$	2, 3, Prop

qed

$\vdash \quad (\mathsf{halt}\ w \wedge f) ; f_1 \wedge (\mathsf{halt}\ w ; g) \supset (\mathsf{halt}\ w \wedge f) ; (f_1 \wedge g)$
HaltAndChopAndHaltChopImpHaltAndChopAnd

Proof:

1	$\vdash$	$f_1 \supset \neg g \vee (f_1 \wedge g)$	Prop
2	$\vdash$	$(\mathsf{halt}\ w \wedge f) ; f_1 \supset$ $(\mathsf{halt}\ w \wedge f) ; \neg g) \vee ((\mathsf{halt}\ w \wedge f) ; (f_1 \wedge g))$	1, ChopOrImpRule
3	$\vdash$	$(\mathsf{halt}\ w \wedge f) ; \neg g \supset \mathsf{halt}\ w ; \neg g$	AndChopA
4	$\vdash$	$\mathsf{halt}\ w ; g \supset \neg(\mathsf{halt}\ w ; \neg g)$	HaltChopImpNotHaltChopNot
5	$\vdash$	$(\mathsf{halt}\ w \wedge f) ; f_1 \wedge (\mathsf{halt}\ w ; g) \supset (\mathsf{halt}\ w \wedge f) ; (f_1 \wedge g)$	2, 3, 4, Prop

qed

$\vdash \quad (\text{halt } w) ; f \supset (\Box \neg w) \mathrel{\text{↝≫}} ((\text{halt } w) ; f)$ **HaltImpBoxYields**

Proof:

1	$\vdash$	$(\Box \neg w) ; \neg(\text{halt } w ; f) \supset \diamondsuit(\Box \neg w)$	**ChopImpDi**
2	$\vdash$	$\Box \neg w \supset \neg w$	**PTL**
3	$\vdash$	$\diamondsuit(\Box \neg w) \supset \diamondsuit \neg w$	2, **DiImpDi**
4	$\vdash$	$\diamondsuit \neg w \equiv \neg w$	**DiState**
5	$\vdash$	$(\Box \neg w) ; \neg(\text{halt } w ; f) \supset \neg w$	1, 2, 4, **Prop**
6	$\vdash$	$\text{halt } w ; f \equiv \text{if } w \text{ then } f \text{ else } \bigcirc(\text{halt } w ; f)$	**HaltChopEqv**
7	$\vdash$	$(\text{halt } w ; f) \wedge (\Box \neg w) ; \neg(\text{halt } w ; f) \supset \bigcirc((\text{halt } w) ; f)$	5, 6, **Prop**
8	$\vdash$	$\Box \neg w \supset \text{empty} \vee \bigcirc \Box \neg w$	**PTL**
9	$\vdash$	$(\Box \neg w) ; \neg(\text{halt } w ; f) \supset$ $\neg(\text{halt } w ; f) \vee \bigcirc((\Box \neg w) ; \neg(\text{halt } w ; f))$	8, **EmptyOrNextChopImpRule**
10	$\vdash$	$(\text{halt } w) ; f \wedge (\Box \neg w) ; \neg(\text{halt } w ; f) \supset$ $\bigcirc((\Box \neg w) ; \neg(\text{halt } w ; f))$	9, **Prop**
11	$\vdash$	$(\text{halt } w) ; f \wedge (\Box \neg w) ; \neg(\text{halt } w ; f) \supset$ $\bigcirc((\text{halt } w) ; f) \wedge \bigcirc((\Box \neg w) ; \neg(\text{halt } w ; f))$	7, 10, **Prop**
12	$\vdash$	$\bigcirc((\text{halt } w) ; f) \wedge \bigcirc((\Box \neg w) ; \neg(\text{halt } w ; f)) \supset$ $\bigcirc(((\text{halt } w) ; f) \wedge ((\Box \neg w) ; \neg(\text{halt } w ; f)))$	**PTL**
13	$\vdash$	$(\text{halt } w) ; f \wedge (\Box \neg w) ; \neg(\text{halt } w ; f) \supset$ $\bigcirc(((\text{halt } w) ; f) \wedge ((\Box \neg w) ; \neg(\text{halt } w ; f)))$	11, 12, **ImpChain**
14	$\vdash$	$\neg((\text{halt } w) ; f \wedge (\Box \neg w) ; \neg(\text{halt } w ; f))$	13, **NextLoop**
15	$\vdash$	$(\text{halt } w) ; f \supset \neg((\Box \neg w) ; \neg(\text{halt } w ; f))$	14, **Prop**
16	$\vdash$	$(\text{halt } w) ; f \supset (\Box \neg w) \mathrel{\text{↝≫}} ((\text{halt } w) ; f)$	15, def. of ↝≫

qed

2.9 Properties of groups of *chops*

$\vdash \quad (f_1 ; \ldots ; f_m) ; (g_1 ; \ldots ; g_n) \equiv f_1 ; \ldots ; f_m ; g_1 ; \ldots ; g_n$ **ChopGroupMerge**

We prove this by induction on m.

Proof for m=1:

1	$\vdash$	$f_1 ; (g_1 ; \ldots ; g_n) \equiv f_1 ; g_1 ; \ldots ; g_n$	def. of $< chop >$

qed

Proof for $m > 1$:

1	$\vdash$	$(f_1 ; f_2 ; \ldots ; f_m) ; (g_1 ; \ldots ; g_n) \equiv$ $f_1 ; ((f_2 ; \ldots ; f_m) ; (g_1 ; \ldots ; g_n))$	**ChopAssoc**
2	$\vdash$	$(f_2 ; \ldots ; f_m) ; (g_1 ; \ldots ; g_n) \equiv$ $f_2 ; \ldots ; f_m ; g_1 ; \ldots ; g_n$	induction hypothesis
3	$\vdash$	$f_1 ; ((f_2 ; \ldots ; f_m) ; (g_1 ; \ldots ; g_n)) \equiv$ $f_1 ; (f_2 ; \ldots ; f_m ; g_1 ; \ldots ; g_n)$	2, **RightChopEqvChop**
4	$\vdash$	$(f_1 ; f_2 ; \ldots ; f_m) ; (g_1 ; \ldots ; g_n) \equiv$ $f_1 ; f_2 ; \ldots ; f_m ; g_1 ; \ldots ; g_n$	1, 3, **EqvChain**

qed

$\vdash \quad (f_{1,1}\,;\ldots;f_{1,l_1})\,;\ldots;(f_{n,1}\,;\ldots;f_{n,l_n}) \equiv$
$f_{1,1}\,;\ldots;f_{1,l_1}\,;\ldots;f_{n,1}\,;\ldots;f_{n,l_n},$ ChopGroupGroupMerge

where there are n groups and for each group $1 \leq i < n$, there are l_i formulas. Proof is by induction on n.

Proof for $n = 1$:

1 $\vdash \quad f_{1,1}\,;\ldots;f_{1,l_1} \equiv f_{1,1}\,;\ldots;f_{1,l_1}$ **Prop**

qed

Proof for $n > 1$:

1 $\vdash \quad (f_{2,1}\,;\ldots;f_{2,l_2})\,;\ldots;(f_{n,1}\,;\ldots;f_{n,l_n}) \equiv$
$f_{2,1}\,;\ldots;f_{2,l_2}\,;\ldots;f_{n,1}\,;\ldots;f_{n,l_n}$ induction hypothesis

2 $\vdash \quad (f_{1,1}\,;\ldots;f_{1,l_1})\,;(f_{2,1}\,;\ldots;f_{2,l_2})\,;\ldots;(f_{n,1}\,;\ldots;f_{n,l_n}) \equiv$
$(f_{1,1}\,;\ldots;f_{1,l_1})\,;(f_{2,1}\,;\ldots;f_{2,l_2}\,;\ldots;f_{n,1}\,;\ldots;f_{n,l_n})$ 1, **LeftChopEqvChop**

3 $\vdash \quad (f_{1,1}\,;\ldots;f_{1,l_1})\,;(f_{2,1}\,;\ldots;f_{2,l_2}\,;\ldots;f_{n,1}\,;\ldots;f_{n,l_n}) \equiv$
$f_{1,1}\,;\ldots;f_{1,l_1}\,;f_{2,1}\,;\ldots;f_{2,l_2}\,;\ldots;f_{n,1}\,;\ldots;f_{n,l_n}$ **ChopGroupMerge**

4 $\vdash \quad (f_{1,1}\,;\ldots;f_{1,l_1})\,;(f_{2,1}\,;\ldots;f_{2,l_2})\,;\ldots;(f_{n,1}\,;f_{n,2}\,;\ldots;f_{n,l_n}) \equiv$
$f_{1,1}\,;\ldots;f_{1,l_1}\,;f_{2,1}\,;\ldots;f_{2,l_2}\,;\ldots;f_{n,1}\,;\ldots;f_{n,l_n}$ 2, 3, **EqvChain**

qed

$\vdash \quad f\,;f\,;\ldots;f \supset f^*$ ChopGroupImpCS

The proof is by induction on the number of <chops>.

Proof when no <chops>:

1 $\vdash \quad f \supset f^*$ **ImpCS**

qed

Proof for at n occurrences of <chops> where $n \geq 1$:

1	$\vdash$	$f\,;\ldots;f \supset f^*$	induction for $n-1$ <chops>
2	$\vdash$	$f\,;f\,;\ldots;f \supset f\,;f^*$	1, **LeftChopImpChop**
3	$\vdash$	$f\,;f^* \supset f^*$	**ChopCSImpCS**
4	$\vdash$	$f\,;f\,;\ldots;f \supset f^*$	2, 3, **ImpChain**

qed

$\vdash f_1 \supset g, \quad \ldots, \quad \vdash f_n \supset g \Rightarrow \vdash (f_1\,;\ldots;f_n) \supset g^*$ MultChopImpCS

Proof:

1	$\vdash$	$f_i \supset g, \quad \text{for } 1 \leq i \leq n$	given
2	$\vdash$	$f_1 ; \ldots ; f_n \supset g ; \ldots ; g$	1, **MultChopImpChop**
3	$\vdash$	$g ; \ldots ; g \supset g^*$	**ChopGroupImpCS**
4	$\vdash$	$f_1 ; \ldots ; f_n \supset g^*$	2, 3, **ImpChain**

qed

$$\vdash w \wedge f \ \supset \ g ; (w_1 \wedge f_1), \quad \vdash w_1 \wedge f_1 \ \supset \ g_1 ; (w_2 \wedge f_2) \quad \Rightarrow \quad \vdash w \wedge f \ \supset \ g ; g_1 ; (w_2 \wedge f_2)$$

NestedChopImpChop

Proof:

1	$\vdash$	$w \wedge f \supset g ; (w_1 \wedge f_1)$	given
2	$\vdash$	$w_1 \wedge f_1 \supset g_1 ; (w_2 \wedge f_2)$	given
3	$\vdash$	$g ; (w_1 \wedge f_1) \supset g ; g_1 ; (w_2 \wedge f_2)$	2, **RightChopImpChop**
4	$\vdash$	$w \wedge f \supset g ; g_1 ; (w_2 \wedge f_2)$	1, 3, **ImpChain**

qed

$$\vdash w_1 \wedge f_1 \ \supset \ g_1 ; (w_2 \wedge f_2), \ldots, \vdash w_{n-1} \wedge f_{n-1} \ \supset \ g_{n-1} ; (w_n \wedge f_n)$$

MultNestedChopImpChop

$$\Rightarrow \quad \vdash w_1 \wedge f_1 \ \supset \ g_1 ; \ldots ; g_{n-1} ; (w_n \wedge g_n)$$

The proof is by induction on n.

Proof for $n = 1$:

1	$\vdash$	$w_1 \wedge f_1 \supset w_1 \wedge f_1$	**Prop**

qed

Proof for $n > 1$:

1	$\vdash$	$w_1 \wedge f_1 \supset g_1 ; (w_2 \wedge f_2)$	given
2	$\vdash$	$w_i \wedge f_i \supset g_i ; (w_{i+1} \wedge f_{i+1}), \quad \text{for each } i\text{: } 1 < i < n$	given
3	$\vdash$	$w_2 \wedge f_2 \supset g_2 ; \ldots ; g_{n-1} ; (w_n \wedge g_n)$	2, induction hypothesis
4	$\vdash$	$w_1 \wedge f_1 \supset g_1 ; g_2 ; \ldots ; g_{n-1} ; (w_n \wedge g_n)$	1, 3, **NestedChopImpChop**

qed

Part II

JANCL proofs

Proofs taken from

Ben Moszkowski. "A Hierarchical Completeness Proof for Propositional Interval Temporal Logic with Finite Time". In: Journal of Applied Non-Classical Logics 14.1–2 (2004), pp. 55–104. url.

3 Propositional Proofs

$\vdash f_1 \supset f_2, \quad ,\ldots, \quad \vdash f_{n-1} \supset f_n \Rightarrow \quad \vdash f_1 \supset f_n$	ImpChain
$\vdash f_1 \equiv f_2, \quad ,\ldots, \quad \vdash f_{n-1} \equiv f_n \Rightarrow \quad \vdash f_1 \equiv f_n$	EqvChain
$\vdash f_1, \vdash f_2, \ldots, \vdash f_n \quad \Rightarrow \quad \vdash g,$	Prop
where the formula $f_1 \wedge f_2 \wedge \ldots f_n \supset g$	
is a substitution instance of a propositional tautology	

4 PITL Axiom System

We now present an axiom system for PITL. Our experience in rigorously developing hundreds of proofs has helped us refine the axioms and convinced us of their utility for a wide range of purposes.

Definition 1 (Tautology) *A tautology is any formula which is a substitution instance of some valid nonmodal propositional formula.*

For example, any PITL formula of the form $\bigcirc f \vee \Diamond g \supset \Diamond g$ is a tautology since it is a substitution instance of the valid nonmodal formula $h_0 \vee h_1 \supset h_1$. It is not hard to show that all tautologies are themselves valid. Intuitively, a formula is a tautology if it does not require any modal reasoning to justify its truth.

4.1 Axioms and Inference Rules for PITL

Our PITL axiom system is given in Table 1. Recall that the symbol $\supset$ is the logical operator *implication* used in formulas. In contrast, the metalogical symbol $\Rightarrow$ denotes the ability to infer a new theorem from other previously deduced ones. The axiom system mainly deals with *chop*, and skip and operators derived from them. Only one axiom is needed for *chop-star*.

The axiom system contains some of the propositional axioms suggested by Rosner and Pnueli but also includes our own axioms and inference rule for the operators $\boxdot$ and *chop-star*. These assist in deducing theorems and derived inference rules for compositional reasoning. The Axiom **Taut** permits using properties of conventional nonmodal logic without proof (recall Definition 1 concerning tautologies). It is possible to omit it and achieve the same results by means of a few "lower-level" axioms and inference rules dealing primarily with nonmodal reasoning.

The axiom system gives nearly equal treatment to initial and terminal subintervals. For example, the Inference Rules **BiGen** and **BoxGen** respectively provide a means to obtain new theorems by embedding

$\vdash$All PITL tautologies	Taut
$\vdash (f\,;g)\,;h \equiv f\,;(g\,;h)$	ChopAssoc
$\vdash (f_0 \vee f_1)\,;g \supset (f_0\,;g) \vee (f_1\,;g)$	OrChopImp
$\vdash f\,;(g_0 \vee g_1) \supset (f\,;g_0) \vee (f\,;g_1)$	ChopOrImp
$\vdash \mathsf{empty}\,;f \equiv f$	EmptyChop
$\vdash f\,;\mathsf{empty} \equiv f$	ChopEmpty
$\vdash w \supset \boxdot\, w$	StateImpBi
$\vdash \boxdot(f_0 \supset f_1) \wedge \Box(g_0 \supset g_1) \supset (f_0\,;g_0) \supset (f_1\,;g_1)$	BiBoxChopImpChop
$\vdash \bigcirc f \supset \textcircled{w} f$	NextImpWeakNext
$\vdash f \wedge \Box(f \supset \textcircled{w} f) \supset \Box f$	BoxInduct
$\vdash f^* \equiv \mathsf{empty} \vee (f \wedge \mathsf{more})\,;f^*$	ChopStarEqv
$\vdash f \supset g, \quad \vdash f \quad \Rightarrow \quad \vdash g$	MP
$\vdash f \quad \Rightarrow \quad \vdash \Box f$	BoxGen
$\vdash f \quad \Rightarrow \quad \vdash \boxdot f$	BiGen

Table 1: PITL axiom system

previously deduced PITL theorems in $\boxdot$ and $\Box$. This is exceedingly important for the kinds of proofs we do since we naturally move formulas in and out of the left side of chop in many situations. The later embedding of the FL axiom system in the PITL axiom system and the reduction of PITL completeness to FL completeness both involve a lot of this kind of reasoning. The proof of the PITL Replacement Theorem is also a good example of how the analysis of the left side of chop is relevant. We additionally believe that axioms and inference rules concerning $\boxdot$ make the axiom system easier to understand since much of it consists simply of duals in this sense. In contrast, most temporal logics cannot readily handle initial subintervals since the conventional operators are point-based. Even other axiom systems for ITL largely neglect initial subintervals.

A formula f which is deducible (provable) from the axioms and inferences rules is called an *PITL theorem*, denoted $\vdash f$. When doing proofs, we can observe that a PITL subset in which the only primitive temporal operator is chop and one side is always some fixed formula obeys the rules of the conventional normal modal system K. We now give two sample theorems and their proofs. The justification **Prop** in some steps refers to conventional propositional reasoning which can involve implicit uses of Axiom **Taut** and/or modus ponens **MP**.

$$\vdash \quad \boxdot(f \supset g) \supset \Diamond\!\!\!\!\cdot\;\, f \supset \Diamond\!\!\!\!\cdot\;\, g \qquad \text{BiImpDiImpDiSample}$$

Proof:

1	$\mathrm{true} \supset \mathrm{true}$	**Prop**
2	$\Box(\mathrm{true} \supset \mathrm{true})$	1, **BoxGen**
3	$\boxdot(f \supset g) \wedge \Box(\mathrm{true} \supset \mathrm{true})$	
	$\supset (f\,;\mathrm{true}) \supset (g\,;\mathrm{true})$	**BiBoxChopImpChop**
4	$\boxdot(f \supset g) \supset (f\,;\mathrm{true}) \supset (g\,;\mathrm{true})$	2, 3, **Prop**
5	$\boxdot(f \supset g) \supset \Diamond\!\!\!\cdot\, f \supset \Diamond\!\!\!\cdot\, g$	4, def. of $\Diamond\!\!\!\cdot$

qed

The following instance of Axiom **StateImpBi** illustrates why it is not subsumed by Inference Rule **BiGen**:

$$\vdash \quad \neg Q \supset \boxdot \neg Q$$

Here Q is a propositional variable. We cannot use **BiGen** since $\neg Q$ is not a theorem.

5 Deduction of PTL Axioms from the FL Axiom System

$\vdash \quad \Box(f \supset g) \supset \Box f \supset \Box g$	A1
$\vdash \quad \bigcirc f \supset \textcircled{w} f$	A2
$\vdash \quad \bigcirc(f \supset g) \supset \bigcirc f \supset \bigcirc g$	A3
$\vdash \quad \Box f \supset f \wedge \textcircled{w}\Box f$	A4
$\vdash \quad \Box(f \supset \textcircled{w} f) \supset f \supset \Box f$	A5
f is a tautology $\quad \Rightarrow \quad \vdash \quad f$	R1
$\vdash \quad f \supset g, \quad \vdash \quad f \quad \Rightarrow \quad \vdash \quad g$	R2
$\vdash \quad f \quad \Rightarrow \quad \vdash \quad \Box f$	R3

Table 2: Modified version of Pnueli's complete axiom system

This appendix contains various FL theorems and their deductions. These include ones corresponding to some of the PTL axioms in Table 2. Most of the PTL axioms and inference rules have identical or nearly identical versions in the FL axiom system in Table 3. The three exceptions are Axioms **A1**, **A3** and **A4**. We will look at each of them in turn as FL theorems **FBoxImpDist**, **FNextImpDist** and **FBoxImpNowAndWeakNext**, respectively. The trickiest is Axiom **A1**. The symbol $\vdash$ as used here always refers to $\vdash_{FL}$. None of the FE formulas occurring in the proofs contain variables and therefore the proofs also ensure well-formed FLV theorems and derived inference rules for any V.

$$\vdash \quad \Box(f \supset g) \supset \Diamond f \supset \Diamond g \qquad \text{FBoxImpDiamondImpDiamond}$$

$\vdash$	All FL tautologies	FLTaut
$\vdash$	$\bigcirc X \equiv \langle \mathsf{skip} \rangle f$	FL2
$\vdash$	$\Diamond f \equiv \langle \mathsf{skip}^* \rangle f$	FL3
$\vdash$	$\langle w? \rangle f \equiv w \wedge f$	FL4
$\vdash$	$\langle E_0 \vee E_1 \rangle f \equiv \langle E_0 \rangle f \vee \langle E_1 \rangle f$	FL5
$\vdash$	$\langle E_0 \,;\, E_1 \rangle f \equiv \langle E_0 \rangle \langle E_1 \rangle f$	FL6
$\vdash$	$\langle E \rangle (f \vee g) \supset \langle E \rangle f \vee \langle E \rangle g$	FL7
$\vdash$	$\langle E^* \rangle f \equiv f \vee \langle E \,;\, E^* \rangle f$	FL8
$\vdash$	$\Box (f \supset g) \supset \langle E \rangle f \supset \langle E \rangle g$	FL9
$\vdash$	$\bigcirc f \supset ⓦ f$	FL10
$\vdash$	$\Box (f \supset ⓦ f) \wedge f \supset \Box f$	FL11
$\vdash$	$\Diamond \mathsf{empty}$	FL12
$\vdash$	$f \supset g, \ \vdash f \ \Rightarrow \ \vdash g$	FLMP
$\vdash$	$f \ \Rightarrow \ \vdash \Box f$	FLBoxGen
$\vdash$	$\langle E_0 \rangle \mathsf{empty} \supset \langle E_0 \rangle \mathsf{empty} \ \Rightarrow \ \vdash \ \langle E_0 \rangle f \supset \langle E_1 \rangle f$	FInf3
$\vdash$	$(\mathsf{more} \wedge \langle E_0 \rangle \mathsf{empty}) \langle E_1 \rangle \mathsf{empty} \ \Rightarrow \ \vdash \ \langle E_0{}^* \rangle f \supset \langle E_1{}^* \rangle f$	FInf4

Table 3: Axiom system for FL

Proof:

1	$\Box (f \supset g) \supset \langle \mathsf{skip}^* \rangle f \supset \langle \mathsf{skip}^* \rangle g$	**FL9**
2	$\Diamond f \equiv \langle \mathsf{skip}^* \rangle f$	**FL3**
3	$\Diamond g \equiv \langle \mathsf{skip}^* \rangle g$	**FL3**
4	$\Box (f \supset g) \supset \Diamond f \supset \Diamond g$	2, 3, **Prop**

qed

The following slightly obscure theorem is used in the proof of **FBoxImpDist**:

$\vdash$	$\Box (\neg g \supset \neg f) \supset \Box f \supset \Box g$	FBoxContraPosImpDist

Proof:

1	$\Box (\neg g \supset \neg f) \supset \Diamond \neg g \supset \Diamond \neg f$	**FBoxImpDiamondImpDiamond**
2	$\Box (\neg g \supset \neg f) \supset \neg \Diamond \neg f \supset \neg \Diamond \neg g$	1, **Prop**
3	$\Box (\neg g \supset \neg f) \supset \Box f \supset \Box g$	2, def. of $\Box$

qed

Below is the proof of PTL Axiom **A1** as FL theorem **FBoxImpDist**. In the final step, **ImpChain** stands for a chain of implications.

$$\vdash \quad \Box(f \supset g) \supset \Box f \supset \Box g \qquad \text{FBoxImpDist}$$

Proof:

1	$(f \supset g) \supset (\neg g \supset \neg f)$	**Prop**
2	$\neg(\neg g \supset \neg f) \supset \neg(f \supset g)$	1, **Prop**
3	$\Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$	2, **FLBoxGen**
4	$\Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$ $\supset \Box(f \supset g) \supset \Box(\neg g \supset \neg f)$	**FBoxContraPosImpDist**
5	$\Box(f \supset g) \supset \Box(\neg g \supset \neg f)$	3, 4, **FLMP**
6	$\Box(\neg g \supset \neg f) \supset \Box f \supset \Box g$	**FBoxContraPosImpDist**
7	$\Box(f \supset g) \supset \Box f \supset \Box g$	5, 6, **ImpChain**

qed

$$\vdash \quad \bigcirc\neg f \supset \neg\bigcirc f \qquad \text{FNextNotImpNotNext}$$

Proof:

1	$\bigcirc f \supset \textcircled{w} f$	**FL10**
2	$\bigcirc f \supset \neg\bigcirc\neg f$	1, def. of $\textcircled{w}$
3	$\bigcirc\neg f \supset \neg\bigcirc f$	2, **Prop**

qed

Here is a proof of PTL Axiom **A3**:

$$\vdash \quad \bigcirc(f \supset g) \supset \bigcirc f \supset \bigcirc g \qquad \text{FNextImpDist}$$

Proof:

1	$\langle \mathsf{skip}\rangle(\neg f \vee g) \supset (\langle \mathsf{skip}\rangle\neg f) \vee (\langle \mathsf{skip}\rangle g)$	**FL7**
2	$\bigcirc(\neg f \vee g) \equiv \langle \mathsf{skip}\rangle(\neg f \vee g)$	**FL2**
3	$\bigcirc\neg f \equiv \langle \mathsf{skip}\rangle\neg f$	**FL2**
4	$\bigcirc g \equiv \langle \mathsf{skip}\rangle g$	**FL2**
5	$\bigcirc(\neg f \vee g) \supset \bigcirc\neg f \vee \bigcirc g$	1–4, **Prop**
6	$\bigcirc\neg f \supset \neg\bigcirc f$	**FNextNotImpNotNext**
7	$\bigcirc(\neg f \vee g) \supset \neg\bigcirc f \vee \bigcirc g$	5, 6, **Prop**
8	$\bigcirc(f \supset g) \supset \bigcirc f \supset \bigcirc g$	7, def. of $\supset$

qed

The remaining proofs are for ultimately deducing PTL Axiom **A4** as FL theorem **FBoxImpNowAndWeakNext**. The following derived rule **FRightSkipChopImpSkipChopRule** can be readily generalised to allow some arbitrary FE formula in place of skip. In addition, a version can be proven which uses $\equiv$ instead of $\supset$.

$$\vdash \quad f \supset g \quad \Rightarrow \quad \vdash \quad \langle \mathsf{skip}\rangle f \supset \langle \mathsf{skip}\rangle g \qquad \text{FRightSkipChopImpSkipChopRule}$$

Proof:

1	$f \supset g$	Assump
2	$\Box(f \supset g)$	**FLBoxGen**
3	$\Box(f \supset g) \supset \langle \text{skip} \rangle f \supset \langle \text{skip} \rangle g$	**FL9**
4	$\langle \text{skip} \rangle f \supset \langle \text{skip} \rangle g$	2, 3, **MP**

qed

$$\vdash \quad f \supset g \quad \Rightarrow \quad \bigcirc f \supset \bigcirc g$$ FNextImpNextRule

Proof:

1	$f \supset g$	Assump
2	$\langle \text{skip} \rangle f \supset \langle \text{skip} \rangle g$	1, **FRightSkipChopImpSkipChopRule**
3	$\bigcirc f \equiv \langle \text{skip} \rangle f$	**FL2**
4	$\bigcirc g \equiv \langle \text{skip} \rangle g$	**FL2**
5	$\bigcirc f \supset \bigcirc g$	3, 4, **Prop**

qed

$$\vdash \quad f \equiv g \quad \Rightarrow \quad \bigcirc f \equiv \bigcirc g$$ FNextEqvNextRule

Proof:

1	$f \equiv g$	Assump
2	$f \supset g$	1, **Prop**
3	$\bigcirc f \supset \bigcirc g$	2, **FNextImpNextRule**
4	$g \supset f$	1, **Prop**
5	$\bigcirc g \supset \bigcirc f$	4, **FNextImpNextRule**
6	$\bigcirc f \equiv \bigcirc g$	3, 5, **Prop**

qed

$$\vdash \quad \Diamond f \equiv f \lor \bigcirc \Diamond f$$ FDiamondEqvNowOrNextDiamond

Proof:

1	$\Diamond f \equiv \langle \text{skip}^* \rangle f$	**FL3**
2	$\langle \text{skip}^* \rangle f \equiv f \lor \langle \text{skip} ; \text{skip}^* \rangle f$	**FL8**
3	$\langle \text{skip} ; \text{skip}^* \rangle f \equiv \langle \text{skip} \rangle \langle \text{skip}^* \rangle f$	**FL6**
4	$\bigcirc \langle \text{skip}^* \rangle f \equiv \langle \text{skip} \rangle \langle \text{skip}^* \rangle f$	**FL2**
5	$\bigcirc \Diamond f \equiv \bigcirc \langle \text{skip}^* \rangle X$	1, **FNextEqvNextRule**
6	$\Diamond f \equiv f \lor \bigcirc \Diamond f$	1–5, **Prop**

qed

$$\vdash \quad f \supset \Diamond f$$ FNowImpDiamond

Proof:

1 $\Diamond f \equiv f \vee \bigcirc\Diamond f$ FDiamondEqvNowOrNextDiamond
2 $f \supset \Diamond f$ 1, Prop
qed

$\vdash \bigcirc\Diamond f \supset \Diamond f$ FNextDiamondImpDiamond

Proof:
1 $\Diamond f \equiv f \vee \bigcirc\Diamond f$ FDiamondEqvNowOrNextDiamond
2 $\bigcirc\Diamond f \supset \Diamond f$ 1, Prop
qed

$\vdash \Box f \supset f$ BoxImpNow

Proof:
1 $\neg f \supset \Diamond\neg f$ FNowImpDiamond
2 $\neg\Diamond\neg f \supset f$ 1, Prop
3 $\Box f \supset f$ 2, def. of $\Box$
qed

$\vdash \Box f \supset ⓦ\Box f$ FBoxImpWeakNextBox

Proof:
1 $\neg\neg\Diamond\neg f \supset \Diamond\neg f$ Prop
2 $\bigcirc\neg\neg\Diamond\neg f \supset \bigcirc\Diamond\neg f$ 1, FNextImpNextRule
3 $\bigcirc\Diamond\neg f \supset \Diamond\neg f$ FNextDiamondImpDiamond
4 $\bigcirc\neg\neg\Diamond\neg f \supset \Diamond\neg f$ 2, 3, ImpChain
5 $\bigcirc\neg\Box f \supset \Diamond\neg f$ 4, def. of $\Box$
6 $\neg\Diamond\neg f \supset \neg\bigcirc\neg\Box f$ 5, Prop
7 $\Box f \supset ⓦ\Box f$ 6, def. of $\Box$, ⓦ
qed

Below is a proof of PTL Axiom **A4**:

$\vdash \Box f \supset f \wedge ⓦ\Box f$ FBoxImpNowAndWeakNext

Proof:
1 $\Box f \supset f$ BoxImpNow
2 $\Box f \supset ⓦ\Box f$ FBoxImpWeakNextBox
3 $\Box f \supset f \wedge ⓦ\Box f$ 1, 2, Prop
qed

Part III

LMCS proofs

Proofs taken from

Ben C. Moszkowski. "A Complete Axiom System for Propositional Interval Temporal Logic with Infinite Time". In: Logical Methods in Computer Science Journal 8.3 (2012). url.

6 Axiom system for PITL with finite and infinite time

$\vdash$Substitution instances of valid PTL formulas	VPTL
$\vdash (f \frown g) \frown h \equiv f \frown (g \frown h)$	ChopAssoc
$\vdash (f_0 \vee f_1) \frown g \supset (f_0 \frown g) \vee (f_1 \frown g)$	OrChopImp
$\vdash f \frown (g_0 \vee g_1) \supset (f \frown g_0) \vee (f \frown g_1)$	ChopOrImp
$\vdash \mathsf{empty} \frown f \equiv f$	EmptyChop
$\vdash \mathsf{finite} \supset (f \frown \mathsf{empty} \equiv f)$	FiniteImpChopEmpty
$\vdash w \supset \boxed{\mathrm{i}}\, w$	StateImpBf
$\vdash \boxed{\mathrm{f}}(f_0 \supset f_1) \wedge \Box(g_0 \supset g_1) \supset (f_0 \frown g_0) \supset (f_1 \frown g_1)$	BfAndBoxImpChopImpChop
$\vdash f^{\star} \equiv \mathsf{empty} \vee (f \wedge \mathsf{more}) \frown f^{\star}$	SChopStarEqv
$\vdash f \wedge \Box(f \supset (g \wedge \mathsf{more}) \frown f) \supset g^{\omega}$	ChopOmegaInduct
$\vdash f \supset g, \quad \vdash f \quad \Rightarrow \quad \vdash g$	MP
$\vdash \mathsf{finite} \supset f \quad \Rightarrow \quad \vdash \boxed{\mathrm{f}}\, f$	BfFGen
$\vdash f \quad \Rightarrow \quad \vdash \Box f$	BoxGen
$\vdash \boxed{\mathrm{f}}((\mathsf{fin}\ P) \equiv g) \supset f \quad \Rightarrow \quad \vdash f$	BfAux

In **BfAux**, propositional variable P must not occur in f and g.

$\vdash f_1 \supset f_2, \quad ,\ldots, \quad \vdash f_{n-1} \supset f_n \Rightarrow \quad \vdash f_1 \supset f_n$	ImpChain
$\vdash f_1 \equiv f_2, \quad ,\ldots, \quad \vdash f_{n-1} \equiv f_n \Rightarrow \quad \vdash f_1 \equiv f_n$	EqvChain
$\vdash f_1, \vdash f_2, \ldots, \vdash f_n \quad \Rightarrow \quad \vdash g,$ where the formula $f_1 \wedge f_2 \wedge \ldots f_n \supset g$ is a substitution instance of a propositional tautology	Prop

7 Axiom system for PITL with finite time

$\vdash$Substitution instances of conventional (nonmodal) tautologies	Taut
$\vdash (f \frown g) \frown h \equiv f \frown (g \frown h)$	FChopAssoc
$\vdash (f_0 \vee f_1) \frown g \supset (f_0 \frown g) \vee (f_1 \frown g)$	FOrChopImp
$\vdash f \frown (g_0 \vee g_1) \supset (f \frown g_0) \vee (f \frown g_1)$	FChopOrImp
$\vdash \mathrm{empty} \frown f \equiv f$	FEmptyChop
$\vdash f \frown \mathrm{empty} \equiv f$	FChopEmpty
$\vdash w \supset \boxed{\mathrm{f}}\, w$	FStateImpBf
$\vdash \boxed{\mathrm{f}}(f_0 \supset f_1) \wedge \Box(g_0 \supset g_1) \supset (f_0 \frown g_0) \supset (f_1 \frown g_1)$	FBfAndBoxImpChopImpChop
$\vdash f^{\star} \equiv \mathrm{empty} \vee (f \wedge \mathrm{more}) \frown f^{\star}$	FChopStarEqv
$\vdash \bigcirc f \supset \textcircled{w} f$	FNextImpWeakNext
$\vdash f \wedge \Box(f \supset \textcircled{w} f) \supset \Box f$	FBoxInduct
$\vdash f \supset g, \quad \vdash f \quad \Rightarrow \quad \vdash g$	FMP
$\vdash f \quad \Rightarrow \quad \vdash \boxed{\mathrm{f}}\, f$	FBfGen
$\vdash f \quad \Rightarrow \quad \vdash \Box f$	FBoxGen

Note: $\textcircled{w} f \triangleq \neg \bigcirc \neg f$ (Weak next)

Table 4: Axiom system for PITL with just finite time

8 Some PITL theorems and Their Proofs

This appendix gives a representative set of PITL theorems and derived inference rules together with their proofs. Many are used either directly or indirectly in the completeness proof for PITL with both finite and infinite time. We have partially organised the material, particularly in Section 8.2, along the lines of some standard modal logic systems.

The PITL theorems and derived rules have a shared index sequence (e.g., **BfChopImpChop** – **BoxChopEqvChop** are followed by **BfGen** rather than **DR1**). We believe that this convention simplifies locating material in this appendix.

Proof steps can refer to axioms, inference rules, previously deduced theorems, derived inference rules and also the following:

- Assumptions which are regarded as being previously deduced.
- **Prop**: Conventional nonmodal propositional reasoning (by restricted application of Axiom **VPTL**) and Modus Ponens.
- **ImpChain**: A chain of implications.
- **EqvChain**: A chain of equivalences.
- In principle, **ImpChain** and **EqvChain** are subsumed by **Prop** but are used here to make the reasoning more explicit.

Our assumption of axiomatic completeness for PITL with just finite time permits any valid implication of the form finite $\supset f$.	PITLF

8.1 Some Basic Properties of Chop

We now consider deducing various simple properties of chop and the associated operators $\diamondplus$, $\boxplus$, $\Diamond$ and $\Box$ which have a wide range of uses.

$\vdash \quad \boxplus(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g)$	BfChopImpChop

Proof:

1	$g \supset g$	**Prop**
2	$\Box(g \supset g)$	1, **BoxGen**
3	$\boxplus(f \supset f_1) \wedge \Box(g \supset g) \supset (f \frown g) \supset (f_1 \frown g)$	**BfAndBoxImpChopImpChop**
4	$\boxplus(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g)$	2, 3, **Prop**

qed

$\vdash \quad \Box(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	BoxChopImpChop

Proof:

1	finite $\supset (f \supset f)$	**Prop**
2	$\boxdot(f \supset f)$	1, **BfFGen**
3	$\boxdot(f \supset f) \wedge \Box(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	**BfAndBoxImpChopImpChop**
4	$\Box(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	2, 3, **Prop**

qed

$$\vdash \quad \Box(g \equiv g_1) \supset (f \frown g) \equiv (f \frown g_1) \qquad \text{BoxChopEqvChop}$$

Proof:

1	$\Box(g \equiv g_1) \equiv \Box(g \supset g_1) \wedge \Box(g_1 \supset g)$	**VPTL**
2	$\Box(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	**BoxChopImpChop**
3	$\Box(g_1 \supset g) \supset (f \frown g_1) \supset (f \frown g)$	**BoxChopImpChop**
4	$\Box(g \equiv g_1) \supset (f \frown g) \equiv (f \frown g_1)$	2, 3, **Prop**

qed

The following derived variant of Inference Rule **BfFGen** omits the subformula finite:

$$\vdash \quad f \quad \Rightarrow \quad \vdash \quad \boxdot f \qquad \text{BfGen}$$

Proof:

1	f	Assump
2	finite $\supset f$	1, **Prop**
3	$\boxdot f$	2, **BfFGen**

qed

The derived inference rule **BfGen** can also be referred to as $\boxdot$**Gen** (analogous to the inference rule **BoxGen**).

$$\vdash \quad f \supset f_1 \quad \Rightarrow \quad \vdash \quad (f \frown g) \supset (f_1 \frown g) \qquad \text{LeftChopImpChop}$$

Proof:

1	$f \supset f_1$	Assump
2	$\boxdot(f \supset f_1)$	1, **BfGen**
3	$\boxdot(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g)$	**BfChopImpChop**
4	$f \frown g \supset f_1 \frown g$	2, 3, **MP**

qed

$$\vdash \quad f \equiv f_1 \quad \Rightarrow \quad \vdash \quad (f \frown g) \equiv (f_1 \frown g) \qquad \text{LeftChopEqvChop}$$

Proof:

1	$f \equiv f_1$	Assump
2	$f \supset f_1$	1, Prop
3	$f \frown g \supset f_1 \frown g$	2, LeftChopImpChop
4	$f_1 \supset f$	1, Prop
5	$f_1 \frown g \supset f \frown g$	4, LeftChopImpChop
6	$f \frown g \equiv f_1 \frown g$	3, 5, Prop

qed

$\vdash \quad f \supset g \quad \Rightarrow \vdash \quad \diamondsuit f \supset \diamondsuit g$ **DfImpDf**

Proof:

1	$f \supset g$	Assump
2	$f \frown \text{true} \supset g \frown \text{true}$	1, LeftChopImpChop
3	$\diamondsuit f \supset \diamondsuit g$	2, def. of $\diamondsuit$

qed

$\vdash \quad f \equiv g \quad \Rightarrow \vdash \quad \diamondsuit f \equiv \diamondsuit g$ **DfEqvDf**

Proof:

1	$f \equiv g$	Assump
2	$f \frown \text{true} \equiv g \frown \text{true}$	1, LeftChopEqvChop
3	$\diamondsuit f \equiv \diamondsuit g$	2, def. of $\diamondsuit$

qed

$\vdash \quad g \supset g_1 \quad \Rightarrow \vdash \quad (f \frown g) \supset (f \frown g_1)$ **RightChopImpChop**

Proof:

1	$g \supset g_1$	Assump
2	$\Box(g \supset g_1)$	BoxGen
3	$\Box(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	BoxChopImpChop
4	$f \frown g \supset f \frown g_1$	2, 3, MP

qed

$\vdash \quad g \equiv g_1 \quad \Rightarrow \vdash \quad (f \frown g) \equiv (f \frown g_1)$ **RightChopEqvChop**

Proof:

1	$g \equiv g_1$	Assump
2	$g \supset g_1$	1, Prop
3	$f \frown g \supset f \frown g_1$	2, RightChopImpChop
4	$g_1 \supset g$	1, Prop
5	$f \frown g_1 \supset f \frown g$	4, RightChopImpChop
6	$f \frown g \equiv f \frown g_1$	3, 5, Prop

qed

$$\vdash \quad f \equiv g \quad \Rightarrow \quad \vdash \quad \Diamond f \equiv \Diamond g$$ DiamondEqvDiamond

Proof:

1	$f \equiv g$	Assump
2	$\mathrm{true} \frown f \equiv \mathrm{true} \frown g$	1, **RightChopEqvChop**
3	$\Diamond f \equiv \Diamond g$	2, def. of $\Diamond$

qed

$$\vdash \quad f \equiv g \quad \Rightarrow \quad \vdash \quad \Box f \equiv \Box g$$ BoxEqvBox

Proof:

1	$f \equiv g$	Assump
2	$\neg f \equiv \neg g$	1, **Prop**
3	$\Diamond \neg f \equiv \Diamond \neg g$	2, **DiamondEqvDiamond**
4	$\neg \Diamond \neg f \equiv \neg \Diamond \neg g$	3, **Prop**
5	$\Box f \equiv \Box g$	4, def. of $\Box$

qed

$$\vdash \quad \Box f \supset g \quad \Rightarrow \quad \vdash \quad \Box f \supset \Box g$$ BoxImpInferBoxImpBox

Proof:

1	$\Box f \supset g$	Assump
2	$\Box(\Box f \supset g)$	1, **BoxGen**
3	$\Box(\Box f \supset g) \supset (\Box f \supset \Box g)$	**VPTL**
4	$\Box f \supset \Box g$	2, 3, **MP**

qed

$$\vdash \quad (f \wedge f_1) \frown g \supset f \frown g$$ AndChopA

Proof:

1	$f \wedge f_1 \supset f$	**Prop**
2	$(f \wedge f_1) \frown g \supset f \frown g$	1, **LeftChopImpChop**

qed

$$\vdash \quad (f \wedge f_1) \frown g \supset f_1 \frown g$$ AndChopB

Proof:

1	$f \wedge f_1 \supset f_1$	**Prop**
2	$(f \wedge f_1) \frown g \supset f_1 \frown g$	1, **LeftChopImpChop**

qed

$\vdash \quad (f \wedge f_1) \frown g \supset (f \frown g) \wedge (f_1 \frown g)$ AndChopImpChopAndChop

Proof:

1	$(f \wedge f_1) \frown g \supset f \frown g$	**AndChopA**
2	$(f \wedge f_1) \frown g \supset f_1 \frown g$	**AndChopB**
3	$(f \wedge f_1) \frown g \supset (f \frown g) \wedge (f_1 \frown g)$	1, 2, **Prop**

qed

$\vdash \quad (f \wedge f_1) \frown g \equiv (f_1 \wedge f) \frown g$ AndChopCommute

Proof:

1	$f \wedge f_1 \equiv f_1 \wedge f$	**Prop**
2	$(f \wedge f_1) \frown g \equiv (f_1 \wedge f) \frown g$	1, **LeftChopEqvChop**

qed

$\vdash \quad (f \vee f_1) \frown g \equiv (f \frown g) \vee (f_1 \frown g)$ OrChopEqv

The proof for $\supset$ is immediate from axiom **OrChopImp**.

Here is the proof for $\subset$:

1	$f \supset f \vee f_1$	**Prop**
2	$f \frown g \supset (f \vee f_1) \frown g$	1, **LeftChopImpChop**
3	$f_1 \supset f \vee f_1$	**Prop**
4	$f_1 \frown g \supset (f \vee f_1) \frown g$	3, **LeftChopImpChop**
5	$(f \frown g) \vee (f \frown g_1) \supset (f \vee f_1) \frown g$	2, 4, **Prop**

qed

$\vdash \quad f \frown g \supset \diamondsuit f$ ChopImpDf

Proof:

1	$g \supset \mathsf{true}$	**Prop**
2	$f \frown g \supset f \frown \mathsf{true}$	1, **RightChopImpChop**
3	$f \frown g \supset \diamondsuit f$	2, def. of $\diamondsuit$

qed

$\vdash \quad \diamondsuit\, \mathsf{empty}$ DfEmpty

Proof:

1	$\mathsf{empty} \frown \mathsf{true} \equiv \mathsf{true}$	**EmptyChop**
2	$\mathsf{empty} \frown \mathsf{true} \supset \diamonddot\, \mathsf{empty}$	**ChopImpDf**
3	$\diamonddot\, \mathsf{empty}$	1, 2, **Prop**

qed

$$\vdash \quad f \frown g \supset \Diamond g \qquad \text{ChopImpDiamond}$$

Proof:

1	$f \supset \mathsf{true}$	**Prop**
2	$f \frown g \supset \mathsf{true} \frown g$	1, **LeftChopImpChop**
3	$f \frown g \supset \Diamond g$	2, def. of $\Diamond$

qed

8.2 Some Properties of $\boxdot$ involving the Modal System K and Axiom D

The two pairs of operators $\Box$ and $\Diamond$ and $\boxdot$ and $\diamonddot$ obey various standard properties of modal logics. Axiom **VPTL** helps streamline reasoning involving $\Box$ and $\Diamond$. The situation with $\boxdot$ and $\diamonddot$ is quite different since they lack a comparable axiom. Therefore, it is especially beneficial to review some conventional modal systems which assist in organising various useful deductions involving $\boxdot$ and $\diamonddot$.

Table 5 summarises some relevant modal systems, various associated axioms and inference rules.

System		Axiom or inference rule	Axiom or rule name
K:		$M f \triangleq \neg L \neg f$	M-def
	plus	$\vdash \quad L(f \supset g) \supset (L f \supset L g)$	K
	plus	$\vdash \quad f \quad \Rightarrow \quad \vdash \quad L f$	N
T:	K plus	$\vdash \quad L f \supset f$	T
S4:	T plus	$\vdash \quad L f \supset L L f$	4
KD4:	K plus 4 and	$\vdash \quad L f \supset M f$	D

Table 5: Some standard modal systems

Within PITL, as in PTL, the operator $\Box$ can be regarded as the conventional unary *necessity* modality L and the operator $\Diamond$ as the dual *possibility* operator M. The two operators together fulfil the requirements of the modal system S4. We do not need to explicitly prove versions of the S4 axioms in Table 5 for $\Box$ and $\Diamond$. Rather, any PITL formula which is a substitution instance of a valid S4 formula involving $\Box$ and $\Diamond$ can be readily deduced using the PITL proof system's Axiom **VPTL**. Similarly, inference rules based on S4 can be obtained with Axiom **VPTL**, Inference Rule **BoxGen** (which corresponds to the inference rule N of S4) and modus ponens. Moreover, the PITL proof system's Axiom **VPTL** permits using *any* PITL formula which is a substitution instance of some valid PTL formula which can also contain the PTL operator $\bigcirc$. In view of all this, we do not give much further consideration to aspects of S4 with $\Box$ and $\Diamond$.

In contrast to $\Box$, the PITL operator $\boxdot$ does not have a comprehensive axiom analogous to **VPTL**. Therefore, we need to explicitly prove in the PITL axiom system various modal properties of $\boxdot$ and its dual $\diamonddot$. If only finite time is allowed, then $\boxdot$ and $\diamonddot$ act as an S4 system. However, $\boxdot$ with infinite time permitted does not fulfil the requirements of S4, or even those of the weaker modal system T, because

Axiom T fails. Instead, $\Box_f$ with infinite time fulfils the requirements of the modal system KD4 which is strictly weaker than S4.

Here is a list of KD4's axioms and inference rules and related PITL proofs for $\Box_f$:

K	$\vdash$	$L(f \supset g) \supset (L\,f \supset L\,g)$	Theorem **BfImpDist**
N	$\vdash$	$f \;\Rightarrow\; \vdash \; L\,f$	Derived Inf. Rule **BfGen**
D	$\vdash$	$L\,f \supset M\,f$	Theorem **BfImpDf**
4	$\vdash$	$L\,f \supset LL\,f$	Theorem **BfImpBfBf**

If only finite time is allowed, then the implication D does not need to be regarded as an explicit axiom since it can be inferred from any proof system for S4.

It is also worth noting that the related operators $\Box_i$ and $\Diamond_i$ obey the modal system S4 even when infinite time is permitted. However, we prefer to work with $\Box_f$ and $\Diamond_f$ since the use of strong chop simplifies the overall PITL completeness proof.

Conventional model logics usually take L, not M, to be primitive. When we deduce standard modal properties for $\Box_f$ and $\Diamond_f$ in our PITL axiom system, we let M, which corresponds to $\Diamond_f$, be primitive and define L to be M's dual (i.e., $L\,A \triangleq \neg M \neg f$). This M-based approach goes well with the PITL axioms for chop. Chellas discusses some alternative axiomatisations of modal systems with M as the primitive although none correspond directly to ours. For the system K, we can deduce implication **LImpMImpM** below for $\Box_f$ and $\Diamond_f$ (see Theorem **BfImpDfImpDf** later on) and then obtain from it together some other reasoning the more standard axiom K just presented which only mentions L:

$$\vdash \quad L(f \supset g) \supset (Mf \supset Mg) \qquad \text{LImpMImpM}$$

The operators $\Box$ and $\Box_f$ together yield a *multi-modal logic* with two necessity constructs L and L' which are commutative:

$$\vdash \quad LL'\,f \equiv L'L\,f$$

This corresponds to our Theorem **BfBoxEqvBoxBf** given later on.

Below are various theorems and derived inference rules about $\Box_f$ and $\Diamond_f$ for obtaining the axioms M-def (Theorem **Mdef**) and K (Theorem **BfImpDist**) found in the modal system K. The associated inference rule N was already proved above as Derived Inference Rule **BfGen**. We also prove the modal axiom D (Theorem **BfImpDf**).

In the next proof's final step, recall that **EqvChain** indicates a chain of equivalences:

$$\vdash \quad \Diamond_f\, f \equiv \neg \Box_f \neg f \qquad \text{Mdef}$$

Proof:

1	$f \equiv \neg\neg f$	**Prop**
2	$\Diamond_f\, f \equiv \Diamond_f \neg\neg f$	1, **DfEqvDf**
3	$\Diamond_f \neg\neg f \equiv \neg\neg \Diamond_f \neg\neg f$	**Prop**
4	$\Diamond_f \neg\neg f \equiv \neg \Box_f \neg f$	3, def. of $\Box_f$
5	$\Diamond_f\, f \equiv \neg \Box_f \neg f$	2, 4, **EqvChain**

qed

$$\vdash \quad \Box_f(f \supset g) \supset \Diamond_f\, f \supset \Diamond_f\, g \qquad \text{BfImpDfImpDf}$$

Proof:

1	$\Box_f(f \supset g) \supset (f \frown \text{true}) \supset (g \frown \text{true})$	**BfChopImpChop**
2	$\Box_f(f \supset g) \supset \Diamond_f\, f \supset \Diamond_f\, g$	1, def. of $\Diamond_f$

qed

$$\vdash \quad \Box_f(\neg g \supset \neg f) \supset (\Box_f\, f) \supset (\Box_f\, g) \qquad \text{BfContraPosImpDist}$$

Proof:

1	$\Box_f(\neg g \supset \neg f) \supset (\Diamond_f \neg g) \supset (\Diamond_f \neg f)$	**BfImpDfImpDf**
2	$\Box_f(\neg g \supset \neg f) \supset (\neg \Diamond_f \neg f) \supset (\neg \Diamond_f \neg g)$	1, **Prop**
3	$\Box_f(\neg g \supset \neg f) \supset (\Box_f\, f) \supset (\Box_f\, g)$	2, def. of $\Box_f$

qed

$$\vdash \quad \Box_f(f \supset g) \supset (\Box_f\, f) \supset (\Box_f\, g) \qquad \text{BfImpDist}$$

Proof:

1	$(f \supset g) \supset (\neg g \supset \neg f)$	**Prop**
2	$\neg(\neg g \supset \neg f) \supset \neg(f \supset g)$	1, **Prop**
3	$\Box_f(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$	2, **BfGen**
4	$\Box_f(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$	
	$\supset \Box_f(f \supset g) \supset \Box_f(\neg g \supset \neg f)$	**BfContraPosImpDist**
5	$\Box_f(f \supset g) \supset \Box_f(\neg g \supset \neg f)$	3, 4, **MP**
6	$\Box_f(\neg g \supset \neg f) \supset (\Box_f\, f) \supset (\Box_f\, g)$	**BfContraPosImpDist**
7	$\Box_f(f \supset g) \supset (\Box_f\, f) \supset (\Box_f\, g)$	5, 6, **ImpChain**

qed

$$\vdash \quad f \supset g \quad \Rightarrow \quad \vdash \quad \Box_f\, f \supset \Box_f\, g \qquad \text{BfImpBfRule}$$

Proof:

1	$f \supset g$	Assump
2	$\Box_f(f \supset g)$	1, **BfGen**
3	$\Box_f(f \supset g) \supset (\Box_f\, f) \supset (\Box_f\, g)$	**BfImpDist**
4	$\Box_f\, f \supset \Box_f\, g$	2, 3, **MP**

qed

$$\vdash \quad f \equiv g \quad \Rightarrow \quad \vdash \quad \Box_f\, f \equiv \Box_f\, g \qquad \text{BfEqvBfRule}$$

Proof:

1	$f \equiv g$	Assump
2	$f \supset g$	1, **Prop**
3	$\boxdot f \supset \boxdot g$	2, **BfImpBfRule**
4	$g \supset f$	1, **Prop**
5	$\boxdot g \supset \boxdot f$	4, **BfImpBfRule**
6	$\boxdot f \equiv \boxdot g$	3, 5, **Prop**

qed

$\vdash \quad \boxdot(f \wedge g) \equiv \boxdot f \wedge \boxdot g$ BfAndEqv

Proof:

1	$(f \wedge g) \supset f$	**Prop**
2	$\boxdot(f \wedge g) \supset \boxdot f$	1, **BfImpBfRule**
3	$(f \wedge g) \supset g$	**Prop**
4	$\boxdot(f \wedge g) \supset \boxdot g$	3, **BfImpBfRule**
5	$f \supset (g \supset (f \wedge g))$	**Prop**
6	$\boxdot f \supset \boxdot(g \supset (f \wedge g))$	5, **BfImpBfRule**
7	$\boxdot(g \supset (f \wedge g)) \supset (\boxdot g \supset \boxdot(f \wedge g))$	**BfImpDist**
8	$\boxdot f \wedge \boxdot g \supset \boxdot(f \wedge g)$	6, 7, **Prop**
9	$\boxdot(f \wedge g) \equiv \boxdot f \wedge \boxdot g$	2, 4, 8, **Prop**

qed

$\vdash \quad \boxdot(f \equiv g) \equiv \boxdot(f \supset g) \wedge \boxdot(g \supset f)$ BfEqvSplit

Proof:

1	$(f \equiv g) \equiv (f \supset g) \wedge (g \supset f)$	**Prop**
2	$\boxdot(f \equiv g) \equiv \boxdot((f \supset g) \wedge (g \supset f))$	1, **BfEqvBfRule**
3	$\boxdot((f \supset g) \wedge (g \supset f)) \equiv \boxdot(f \supset g) \wedge \boxdot(g \supset f)$	**BfAndEqv**
4	$\boxdot(f \equiv g) \equiv \boxdot(f \supset g) \wedge \boxdot(g \supset f)$	2, 3, **EqvChain**

qed

$\vdash \quad \boxdot(f \equiv f_1) \supset (f \frown g) \equiv (f_1 \frown g)$ BfChopEqvChop

Proof:

1	$\boxdot(f \equiv f_1) \equiv \boxdot(f \supset f_1) \wedge \boxdot(f_1 \supset f)$	**BfEqvSplit**
2	$\boxdot(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g)$	**BfChopImpChop**
3	$\boxdot(f_1 \supset f) \supset (f_1 \frown g) \supset (f \frown g)$	**BfChopImpChop**
4	$\boxdot(f \equiv f_1) \supset (f \frown g) \equiv (f_1 \frown g)$	$1 - -3$, **Prop**

qed

$\vdash \quad \boxdot(f \equiv g) \supset \diamondsuit f \equiv \diamondsuit g$ BfImpDfEqvDf

Proof:

1	$\boxdot(f \equiv g) \supset (f \frown \text{true}) \equiv (g \frown \text{true})$	BfChopEqvChop
2	$\boxdot(f \equiv g) \supset \diamondsuit f \equiv \diamondsuit g$	1, def. of $\diamondsuit$

qed

$$\vdash \quad \text{finite} \supset (f \equiv g) \quad \Rightarrow \quad \vdash \quad \diamondsuit f \equiv \diamondsuit g$$ FiniteImpDfEqvDfRule

Proof:

1	$\text{finite} \supset (f \equiv g)$	Assump
2	$\boxdot(f \equiv g)$	1, BfFGen
3	$\boxdot(f \equiv g) \supset \diamondsuit f \equiv \diamondsuit g$	BfImpDfEqvDf
4	$\diamondsuit f \equiv \diamondsuit g$	2, 3, MP

qed

$$\vdash \quad \boxdot f \supset \diamondsuit f$$ BfImpDf

Proof:

1	$f \supset (\text{empty} \supset f)$	Prop
2	$\boxdot f \supset \boxdot(\text{empty} \supset f)$	1, BfImpBfRule
3	$\boxdot(\text{empty} \supset f) \supset (\diamondsuit \text{empty} \supset \diamondsuit f)$	BfImpDfImpDf
4	$\boxdot f \supset (\diamondsuit \text{empty} \supset \diamondsuit f)$	2, 3, ImpChain
5	$\diamondsuit \text{empty}$	DfEmpty
6	$\boxdot f \supset \diamondsuit f$	4, 5, Prop

qed

$$\vdash \quad \diamondsuit(f \lor g) \equiv \diamondsuit f \lor \diamondsuit g$$ DfOrEqv

Proof:

1	$(f \lor g) \frown \text{true} \equiv (f \frown \text{true}) \lor (g \frown \text{true})$	OrChopEqv
2	$\diamondsuit(f \lor g) \equiv \diamondsuit f \lor \diamondsuit g$	1, def. of $\diamondsuit$

qed

$$\vdash \quad \boxdot f \land (f_1 \frown g) \supset (f \land f_1) \frown g$$ BfAndChopImport

Proof:

1	$f \supset (f_1 \supset f \land f_1)$	Prop
2	$\boxdot f \supset \boxdot(f_1 \supset f \land f_1)$	1, BfImpBfRule
3	$\boxdot(f_1 \supset f \land f_1) \supset (f_1 \frown g) \supset (f \land f_1) \frown g$	BfChopImpChop
4	$\boxdot f \land (f_1 \frown g) \supset (f \land f_1) \frown g$	2, 3, Prop

qed

8.3 Some Properties of Chop, $\Diamond_f$ and $\Box_f$ with State Formulas

$\vdash \quad \Diamond_f\, w \equiv w$ DfState

Proof for $\supset$:

1	$\neg w \supset \Box_f \neg w$	StateImpBf
2	$\neg w \supset \neg \Diamond_f \neg\neg w$	1, def. of $\Box_f$
3	$\Diamond_f \neg\neg w \supset w$	2, Prop
4	$w \supset \neg\neg w$	Prop
5	$\Diamond_f\, w \supset \Diamond_f \neg\neg w$	4, DfImpDf
6	$\Diamond_f\, w \supset w$	3, 5, ImpChain

qed

Proof for $\subset$:

1	$w \supset \Box_f\, w$	StateImpBf
2	$\Box_f\, w \supset \Diamond_f\, w$	BfImpDf
3	$w \supset \Diamond_f\, w$	1, 2, ImpChain

qed

$\vdash \quad \Box_f\, w \equiv w$ BfState

Proof:

1	$\Diamond_f \neg w \equiv \neg w$	DfState
2	$\neg \Diamond_f \neg w \equiv w$	1, Prop
3	$\Box_f\, w \equiv w$	2, def. of $\Box_f$

qed

$\vdash \quad w \frown f \supset w$ StateChop

Proof:

1	$w \frown f \supset \Diamond_f\, w$	ChopImpDf
2	$\Diamond_f\, w \equiv w$	DfState
3	$w \frown f \supset w$	1, 2, Prop

qed

$\vdash \quad (w \wedge f) \frown g \supset w$ StateChopExportA

Proof:

1	$w \wedge f \supset w$	Prop
2	$(w \wedge f) \frown g \supset w \frown g$	1, LeftChopImpChop
3	$w \frown g \supset w$	StateChop
4	$(w \wedge f) \frown g \supset w$	2, 3, ImpChain

qed

The following lets us move a state formula into the left side of chop:

$$\vdash \quad w \wedge (f \frown g) \supset (w \wedge f) \frown g \qquad \text{StateAndChopImport}$$

Proof:

1	$w \supset \boxdot w$	**StateImpBf**
2	$w \wedge (f \frown g) \supset \boxdot w \wedge (f \frown g)$	1, **Prop**
3	$\boxdot w \wedge (f \frown g) \supset (w \wedge f) \frown g$	**BfAndChopImport**
4	$w \wedge (f \frown g) \supset (w \wedge f) \frown g$	2, 3, **ImpChain**

qed

We can easily combine this with theorem **StateChopExportA** to deduce the equivalence below:

$$\vdash \quad (w \wedge f) \frown g \equiv w \wedge (f \frown g) \qquad \text{StateAndChop}$$

Proof:

1	$(w \wedge f) \frown g \supset w$	**StateChopExportA**
2	$(w \wedge f) \frown g \supset (w \frown g) \wedge (f \frown g)$	**AndChopImpChopAndChop**
3	$(w \wedge f) \frown g \supset w \wedge (f \frown g)$	1, 2, **Prop**
4	$w \wedge (f \frown g) \supset (w \wedge f) \frown g$	**StateAndChopImport**
5	$w \wedge (f \frown g) \equiv (w \wedge f) \frown g$	3, 4, **Prop**

qed

Below is a useful corollary of **StateAndChop** used in decomposing the left side of chop:

$$\vdash \quad (w \wedge \text{empty}) \frown f \equiv w \wedge f \qquad \text{StateAndEmptyChop}$$

Proof:

1	$(w \wedge \text{empty}) \frown f \equiv w \wedge (\text{empty} \frown f)$	**StateAndChop**
2	$\text{empty} \frown f \equiv f$	**EmptyChop**
3	$(w \wedge \text{empty}) \frown f \equiv w \wedge f$	1, 2, **Prop**

qed

The following is a simple corollary of **StateAndEmptyChop**:

$$\vdash \quad (\text{empty} \wedge w) \frown f \equiv w \wedge f \qquad \text{EmptyAndStateChop}$$

Proof:

1	$(\text{empty} \wedge w) \frown f \equiv (w \wedge \text{empty}) \frown f$	**AndChopCommute**
2	$(w \wedge \text{empty}) \frown f \equiv w \wedge f$	**StateAndEmptyChop**
3	$(\text{empty} \wedge w) \frown f \equiv w \wedge f$	1, 2, **EqvChain**

qed

$\vdash \quad \Diamond_{\mathrm{f}}(w \wedge f) \equiv w \wedge \Diamond_{\mathrm{f}}\, f$ StateAndDf

Proof:

1	$(w \wedge f) \frown \mathrm{true} \equiv w \wedge (f \frown \mathrm{true})$	**StateAndChop**
2	$\Diamond_{\mathrm{f}}(w \wedge f) \equiv w \wedge \Diamond_{\mathrm{f}}\, f$	1, def. of $\Diamond_{\mathrm{f}}$

qed

$\vdash \quad w \supset f \quad \Rightarrow \quad \vdash \quad w \supset \Box_{\mathrm{f}}\, f$ StateImpBfGen

Proof:

1	$w \supset f$	Assump
2	$\neg f \supset \neg w$	1, **Prop**
3	$\Diamond_{\mathrm{f}} \neg f \supset \Diamond_{\mathrm{f}} \neg w$	2, **DfImpDf**
4	$\Diamond_{\mathrm{f}} \neg w \equiv \neg w$	**DfState**
5	$\Diamond_{\mathrm{f}} \neg f \supset \neg w$	3, 4, **Prop**
6	$w \supset \neg \Diamond_{\mathrm{f}} \neg f$	5, **Prop**
7	$w \supset \Box_{\mathrm{f}}\, f$	6, def. of $\Box_{\mathrm{f}}$

qed

The following theorem can be used to do induction over time with chop:

$\vdash \quad f \frown g \wedge \neg(f \frown g_1) \supset f \frown (g \wedge \neg g_1)$ ChopAndNotChopImp

Proof:

1	$g \supset (g \wedge \neg g_1) \vee g_1$	**Prop**
2	$f \frown g \supset f \frown (g \wedge \neg g_1) \vee f \frown g_1$	1, **LeftChopImpChop**
3	$f \frown g \wedge \neg(f \frown g_1) \supset f \frown (g \wedge \neg g_1)$	2, **Prop**

qed

8.4 Some Properties of $\Box_{\mathrm{f}}$ involving the Modal System K4

We now consider how to establish for the PITL operator $\Box_{\mathrm{f}}$ the axiom "4" (PITL Theorem **BfImpBfBf**) found in the modal systems *K4* and S4.

$\vdash \quad \Diamond_{\mathrm{f}} \Diamond_{\mathrm{f}}\, f \equiv \Diamond_{\mathrm{f}}\, f$ DfDfEqvDf

Proof:

1	$(f \frown \mathrm{true}) \frown \mathrm{true} \equiv f \frown (\mathrm{true} \frown \mathrm{true})$	**ChopAssoc**
2	$\Diamond_{\mathrm{f}}\, \mathrm{true} \equiv \mathrm{true}$	**DfState**
3	$(\mathrm{true} \frown \mathrm{true}) \equiv \mathrm{true}$	2, def. of $\Diamond_{\mathrm{f}}$
4	$f \frown (\mathrm{true} \frown \mathrm{true}) \equiv f \frown \mathrm{true}$	3, **LeftChopEqvChop**
5	$(f \frown \mathrm{true}) \frown \mathrm{true} \equiv f \frown \mathrm{true}$	1, 4, **EqvChain**
6	$\Diamond_{\mathrm{f}} \Diamond_{\mathrm{f}}\, f \equiv \Diamond_{\mathrm{f}}\, f$	5, def. of $\Diamond_{\mathrm{f}}$

qed

$\vdash \quad \Diamond_f \neg f \equiv \neg \Box_f f$ DfNotEqvNotBf

Proof:

1 $\Box_f f \equiv \neg \Diamond_f \neg f$ def. of $\Box_f$
2 $\Diamond_f \neg f \equiv \neg \Box_f f$ 1, **Prop**

qed

$\vdash \quad \Diamond_f \Diamond_f \neg f \equiv \neg \Box_f \Box_f f$ DfDfNotEqvNotBfBf

Proof:

1 $\Diamond_f \neg f \equiv \neg \Box_f f$ **DfNotEqvNotBf**
2 $\Diamond_f \Diamond_f \neg f \equiv \Diamond_f \neg \Box_f f$ 1, **DfEqvDf**
3 $\Diamond_f \neg \Box_f f \equiv \neg \Box_f \Box_f f$ **DfNotEqvNotBf**
4 $\Diamond_f \Diamond_f \neg f \equiv \neg \Box_f \Box_f f$ 2, 3, **EqvChain**

qed

$\vdash \quad \Box_f \Box_f f \equiv \Box_f f$ BfBfEqvBf

Proof:

1 $\Diamond_f \Diamond_f \neg f \equiv \Diamond_f \neg f$ **DfDfEqvDf**
2 $\Diamond_f \Diamond_f \neg f \equiv \neg \Box_f \Box_f f$ **DfDfNotEqvNotBfBf**
3 $\neg \Box_f \Box_f f \equiv \Diamond_f \neg f$ 1, 2, **Prop**
4 $\Diamond_f \neg f \equiv \neg \Box_f f$ **DfNotEqvNotBf**
5 $\neg \Box_f \Box_f f \equiv \neg \Box_f f$ 3, 4, **EqvChain**
6 $\Box_f \Box_f f \equiv \Box_f f$ 5, **Prop**

qed

$\vdash \quad \Box_f f \supset \Box_f \Box_f f$ BfImpBfBf

Proof:

1 $\Box_f \Box_f f \equiv \Box_f f$ **BfBfEqvBf**
2 $\Box_f f \supset \Box_f \Box_f f$ 1, **Prop**

qed

8.5 Properties Involving the PTL Operator $\bigcirc$

$\vdash \quad (\bigcirc f) \frown g \equiv \bigcirc(f \frown g)$ NextChop

Proof:
1 $(\mathsf{skip} \frown f) \frown g \equiv \mathsf{skip} \frown (f \frown g)$ **ChopAssoc**
2 $(\bigcirc f) \frown g \equiv \bigcirc(f \frown g)$ 1, def. of $\bigcirc$
qed

$\vdash \quad (w \wedge \bigcirc f) \frown g \equiv w \wedge \bigcirc(f \frown g)$ StateAndNextChop

Proof:
1 $(w \wedge \bigcirc f) \frown g \equiv w \wedge ((\bigcirc f) \frown g)$ **StateAndChop**
2 $(\bigcirc f) \frown g \equiv \bigcirc(f \frown g)$ **NextChop**
3 $(w \wedge \bigcirc f) \frown g \equiv w \wedge \bigcirc(f \frown g)$ 1, 2, **Prop**
qed

$\vdash \quad \rlap{\,{\scriptscriptstyle f}}\Diamond(w \wedge \bigcirc w') \equiv w \wedge \bigcirc w'$ DfStateAndNextEqv

Proof:
1 $(w \wedge \bigcirc w') \frown \mathsf{true} \equiv w \wedge \bigcirc(w' \frown \mathsf{true})$ **StateAndNextChop**
2 $\rlap{\,{\scriptscriptstyle f}}\Diamond(w \wedge \bigcirc w') \equiv w \wedge \bigcirc \rlap{\,{\scriptscriptstyle f}}\Diamond\, w'$ 1, def. of $\rlap{\,{\scriptscriptstyle f}}\Diamond$
3 $\rlap{\,{\scriptscriptstyle f}}\Diamond\, w' \equiv w'$ **DfState**
4 $\mathsf{skip} \frown \rlap{\,{\scriptscriptstyle f}}\Diamond\, w' \equiv \mathsf{skip} \frown w'$ 3, **RightChopEqvChop**
5 $\bigcirc \rlap{\,{\scriptscriptstyle f}}\Diamond\, w' \equiv \bigcirc w'$ 4, def. of $\bigcirc$
6 $\rlap{\,{\scriptscriptstyle f}}\Diamond(w \wedge \bigcirc w') \equiv w \wedge \bigcirc w'$ 2, 5, **Prop**
qed

8.6 Some Properties of $\rlap{\,{\scriptscriptstyle f}}\Box$ Together with $\Box$

We make use of the following analogue of Theorem **DfNotEqvNotBf** for $\Diamond$ and $\Box$:

$\vdash \quad \Diamond \neg f \equiv \neg \Box f$ DiamondNotEqvNotBox

Proof:
1 $\Diamond \neg f \equiv \neg \Box f$ **VPTL**
qed

$\vdash \quad \rlap{\,{\scriptscriptstyle f}}\Diamond \Diamond f \equiv \Diamond \rlap{\,{\scriptscriptstyle f}}\Diamond f$ DfDiamondEqvDiamondDf

Proof:
1 $(\mathsf{true} \frown f) \frown \mathsf{true} \equiv \mathsf{true} \frown (f \frown \mathsf{true})$ **ChopAssoc**
2 $(\Diamond f) \frown \mathsf{true} \equiv \Diamond(f \frown \mathsf{true})$ 1, def. of $\Diamond$
3 $\rlap{\,{\scriptscriptstyle f}}\Diamond \Diamond f \equiv \Diamond \rlap{\,{\scriptscriptstyle f}}\Diamond f$ 2, def. of $\rlap{\,{\scriptscriptstyle f}}\Diamond$

qed

$\vdash \quad \Diamond_{\mathrm{f}} \Diamond \neg f \equiv \neg \Box_{\mathrm{f}} \Box f$ DfDiamondNotEqvNotBfBox

Proof:

1	$\Diamond \neg f \equiv \neg \Box f$	DiamondNotEqvNotBox
2	$\Diamond_{\mathrm{f}} \Diamond \neg f \equiv \Diamond_{\mathrm{f}} \neg \Box f$	1, DfEqvDf
3	$\Diamond_{\mathrm{f}} \neg \Box f \equiv \neg \Box_{\mathrm{f}} \Box f$	DfNotEqvNotBf
4	$\Diamond_{\mathrm{f}} \Diamond \neg f \equiv \neg \Box_{\mathrm{f}} \Box f$	2, 3, EqvChain

qed

$\vdash \quad \Diamond \Diamond_{\mathrm{f}} \neg f \equiv \neg \Box \Box_{\mathrm{f}} f$ DiamondDfNotEqvNotBoxBf

Proof:

1	$\Diamond_{\mathrm{f}} \neg f \equiv \neg \Box_{\mathrm{f}} f$	DfNotEqvNotBf
2	$\Diamond \Diamond_{\mathrm{f}} \neg f \equiv \Diamond \neg \Box_{\mathrm{f}} f$	1, DiamondEqvDiamond
3	$\Diamond \neg \Box_{\mathrm{f}} f \equiv \neg \Box \Box_{\mathrm{f}} f$	DiamondNotEqvNotBox
4	$\Diamond \Diamond_{\mathrm{f}} \neg f \equiv \neg \Box \Box_{\mathrm{f}} f$	2, 3, EqvChain

qed

$\vdash \quad \Box_{\mathrm{f}} \Box f \equiv \Box \Box_{\mathrm{f}} f$ BfBoxEqvBoxBf

Proof:

1	$\Diamond_{\mathrm{f}} \Diamond \neg f \equiv \Diamond \Diamond_{\mathrm{f}} \neg f$	DfDiamondEqvDiamondDf
2	$\Diamond_{\mathrm{f}} \Diamond \neg f \equiv \neg \Box_{\mathrm{f}} \Box f$	DfDiamondNotEqvNotBfBox
3	$\Diamond \Diamond_{\mathrm{f}} \neg f \equiv \neg \Box \Box_{\mathrm{f}} f$	DiamondDfNotEqvNotBoxBf
4	$\Box_{\mathrm{f}} \Box f \equiv \Box \Box_{\mathrm{f}} f$	$1-3$, Prop

qed

8.7 Some Properties of Chop-Star

We now consider some theorems and derived rules concerning chop-star.

$\vdash \quad f \supset \text{more} \quad \Rightarrow \quad \vdash \quad f^\star \equiv \text{empty} \vee (f \frown f^\star)$ ImpMoreChopStarEqvRule

Proof:

1	$f \supset \text{more}$	Assump
2	$f \wedge \text{more} \equiv f$	1, Prop
3	$(f \wedge \text{more}) \frown f^\star \equiv f \frown f^\star$	2, LeftChopEqvChop
4	$f^\star \equiv \text{empty} \vee ((f \wedge \text{more}) \frown f^\star)$	SChopStarEqv
5	$f^\star \equiv \text{empty} \vee (f \frown f^\star)$	3, 4, Prop

qed

$\vdash \quad f \supset \text{more} \quad \Rightarrow \quad \vdash \quad f^\star \frown g \equiv g \lor (f \frown (f^\star \frown g))$ ImpMoreChopStarChopEqvRule

Proof:

1	$f \supset \text{more}$	Assump
2	$f^\star \equiv \text{empty} \lor (f \frown f^\star)$	1, **ImpMoreChopStarEqvRule**
3	$f^\star \frown g \equiv (\text{empty} \lor (f \frown f^\star)) \frown g$	2, **LeftChopEqvChop**
4	$(\text{empty} \lor (f \frown f^\star)) \frown g \equiv (\text{empty} \frown g) \lor ((f \frown f^\star) \frown g)$	**OrChopEqv**
5	$\text{empty} \frown g \equiv g$	**EmptyChop**
6	$(f \frown f^\star) \frown g \equiv f \frown (f^\star \frown g)$	**ChopAssoc**
7	$f^\star \frown g \equiv g \lor (f \frown (f^\star \frown g))$	$3 - -6$, **Prop**

qed

$\vdash \quad f^\star \equiv (f^\star \frown \text{empty}) \lor f^\omega$ SChopStarEqvSChopstarChopEmptyOrChopOmega

Proof:

1	$\text{finite} \lor \neg\text{finite}$	**Prop**
2	$\text{finite} \lor \text{inf}$	1, def. of inf
3	$\text{finite} \supset (f^\star \frown \text{empty}) \equiv f^\star$	**FiniteImpChopEmpty**
4	$\text{inf} \supset f^\star \equiv (f^\star \land \text{inf})$	**Prop**
5	$\text{inf} \supset f^\star \equiv f^\omega$	4, def. of chop-omega
6	$f^\star \equiv (f^\star \frown \text{empty}) \lor f^\omega$	2, 3, 5, **Prop**

qed

8.8 Some Properties Involving a Reduction to PITL with Finite Time

We now present some derived inference rules which come in useful when completeness for PITL with finite time is assumed. Recall that any valid implication of the form finite $\supset f$ is allowed and that we designate such a step by using **PITLF**. PITL Theorem **BfFinStateEqvBox** below illustrates this technique.

$\vdash \quad \text{finite} \supset (f \supset g) \quad \Rightarrow \quad \vdash \quad \boxed{f}\, f \supset \boxed{f}\, g$ FiniteImpBfImpBfRule

Proof:

1	$\text{finite} \supset (f \supset g)$	Assump
2	$\boxed{f}(f \supset g)$	1, **BfFGen**
3	$\boxed{f}(f \supset g) \supset (\boxed{f}\, f \supset \boxed{f}\, g)$	**BfImpDist**
4	$\boxed{f}\, f \supset \boxed{f}\, g$	2, 3, **MP**

qed

$$\vdash \quad \text{finite} \supset (f \equiv g) \quad \Rightarrow \quad \vdash \quad \boxed{\scriptstyle f}\, f \equiv \boxed{\scriptstyle f}\, g$$ FiniteImpBfEqvBfRule

Proof:

1	finite $\supset (f \equiv g)$	Assump
2	finite $\supset (f \supset g)$	1, **Prop**
3	$\boxed{\scriptstyle f}\, f \supset \boxed{\scriptstyle f}\, g$	2, **FiniteImpBfImpBfRule**
4	finite $\supset (g \supset f)$	1, **Prop**
5	$\boxed{\scriptstyle f}\, g \supset \boxed{\scriptstyle f}\, f$	4, **FiniteImpBfImpBfRule**
6	$\boxed{\scriptstyle f}\, f \equiv \boxed{\scriptstyle f}\, g$	3, 5, **Prop**

qed

The next theorem's proof involves the application of the previous derived inference rule together with completeness for PITL with just finite time:

$$\vdash \quad \boxed{\scriptstyle f}\, \text{fin } w \equiv \Box\, w$$ BfFinStateEqvBox

Proof:

1	$\boxed{\scriptstyle f}\, \boxed{\scriptstyle f}\, \text{fin } w \equiv \boxed{\scriptstyle f}\, \text{fin } w$	**BfBfEqvBf**
2	$\boxed{\scriptstyle f}\, \text{fin } w \equiv \boxed{\scriptstyle f}\, \boxed{\scriptstyle f}\, \text{fin } w$	1, **Prop**
3	finite $\supset ((\boxed{\scriptstyle f}\, \text{fin } w) \equiv \Box\, w)$	**PITLF**
4	$\boxed{\scriptstyle f}\, \boxed{\scriptstyle f}\, \text{fin } w \equiv \boxed{\scriptstyle f}\, \Box\, w$	3, **FiniteImpBfEqvBfRule**
5	$\boxed{\scriptstyle f}\, \Box\, w \equiv \Box\, \boxed{\scriptstyle f}\, w$	**BfBoxEqvBoxBf**
6	$\boxed{\scriptstyle f}\, w \equiv w$	**BfState**
7	$\Box\, \boxed{\scriptstyle f}\, w \equiv \Box\, w$	6, **BoxEqvBox**
8	$\boxed{\scriptstyle f}\, \text{fin } w \equiv \Box\, w$	2, 4, 5, 7, **EqvChain**

qed

An alternative proof of Theorem **BfFinStateEqvBox** can be given without **PITLF** by first deducing the dual equivalence $(\diamondsuit_f \Diamond(\text{empty} \wedge w)) \equiv \Diamond\, w$, for any state formula w.

8.9 Some Properties of Skip, Next And Until

Recall that NL[1] formulas are exactly those PTL formulas in which the only temporal operators are unnested $\bigcirc$s (e.g., $P \vee \bigcirc\neg P$ but not $P \vee \bigcirc\bigcirc\neg P$). The next theorem holds for any NL[1] formula T:

$$\vdash \quad \diamondsuit_f(\text{more} \wedge T) \equiv \text{more} \wedge T$$ DfMoreAndNLoneEqvMoreAndNLone

Proof 1 *We use Axiom* **VPTL** *to re-express* more $\wedge\, T$ *as a logically equivalent disjunction* $\bigvee_{1\le i\le n}(w_i \wedge \bigcirc w_i')$ *for some natural number* $n \ge 1$ *and* n *pairs of state formulas* w_i *and* w_i'*:*

$$\vdash \quad \text{more} \wedge T \equiv \bigvee_{1\le i\le n} (w_i \wedge \bigcirc w_i')$$ DfMoreAndNLoneEqvMoreAndNLone-1-eq

Now by Theorem ***DfStateAndNextEqv*** *any conjunction* $w \wedge \bigcirc w'$ *is deducibly equivalent to* $\Diamond_f(w \wedge \bigcirc w')$*. Therefore the disjunction in* ***DfMoreAndNLoneEqvMoreAndNLone-1-eq*** *can be re-expressed as* $\bigvee_{1\le i\le n} \Diamond_f(w_i \wedge \bigcirc w_i')$*:*

$$\vdash \quad \bigvee_{1\le i\le n} (w_i \wedge \bigcirc w_i') \equiv \bigvee_{1\le i\le n} \Diamond_f(w_i \wedge \bigcirc w_i') \qquad \text{DfMoreAndNLoneEqvMoreAndNLone-2-eq}$$

Then by $n-1$ *applications of Theorem* ***DfOrEqv*** *and some simple propositional reasoning, the righthand operand of this equivalence is itself is deducibly equivalent to* $\Diamond_f(\bigvee_{1\le i\le n}(w_i \wedge \bigcirc w_i'))$*:*

$$\vdash \quad \bigvee_{1\le i\le n} \Diamond_f(w_i \wedge \bigcirc w_i') \equiv \Diamond_f(\bigvee_{1\le i\le n} (w_i \wedge \bigcirc w_i')) \quad \text{DfMoreAndNLoneEqvMoreAndNLone-3-eq}$$

The chain of the three equivalences
DfMoreAndNLoneEqvMoreAndNLone-1-eq*,*
DfMoreAndNLoneEqvMoreAndNLone-2-eq*, and*
DfMoreAndNLoneEqvMoreAndNLone-3-eq
yields the following:

$$\vdash \quad \text{more} \wedge T \equiv \Diamond_f(\bigvee_{1\le i\le n} (w_i \wedge \bigcirc w_i'))$$

We then apply Derived Rule ***DfEqvDf*** *to the first equivalence* ***DfMoreAndNLoneEqvMoreAndNLone-1-eq****:*

$$\vdash \quad \Diamond_f(\text{more} \wedge T) \equiv \Diamond_f(\bigvee_{1\le i\le n} (w_i \wedge \bigcirc w_i'))$$

The last two equivalences with simple propositional reasoning yield our goal ***DfMoreAndNLoneEqv-MoreAndNLone****.*

Here is a corollary of the previous PITL Theorem **DfMoreAndNLoneEqvMoreAndNLone** for any NL^1 formula T:

$$\vdash \quad \Box_f(\text{more} \supset T) \equiv \text{more} \supset T \qquad \text{BfMoreImpNLoneEqvMoreImpNLone}$$

Proof:

1	$\Box_f(\text{more} \supset T) \equiv \neg \Diamond_f \neg(\text{more} \supset T)$	def. of $\Box_f$
2	$\neg(\text{more} \supset T) \equiv \text{more} \wedge \neg T$	**Prop**
3	$\Diamond_f \neg(\text{more} \supset T) \equiv \Diamond_f(\text{more} \wedge \neg T)$	2, **DfEqvDf**
4	$\Diamond_f(\text{more} \wedge \neg T) \equiv \text{more} \wedge \neg T$	**DfMoreAndNLoneEqvMoreAndNLone**
5	$\Diamond_f \neg(\text{more} \supset T) \equiv \text{more} \wedge \neg T$	3, 4, **EqvChain**
6	$\Box_f(\text{more} \supset T) \equiv \neg(\text{more} \wedge \neg T)$	1, 5, **Prop**
7	$\neg(\text{more} \wedge \neg T) \equiv \text{more} \supset T$	**Prop**
8	$\Box_f(\text{more} \supset T) \equiv \text{more} \supset T$	6, 7, **EqvChain**

qed

$\vdash \quad \text{more} \wedge T \supset \Box_f(\text{more} \supset T)$ MoreAndNLoneImpBfMoreImpNLone

Proof:

1	$\Box_f(\text{more} \supset T) \equiv \text{more} \supset T$	**BfMoreImpNLoneEqvMoreImpNLone**
2	$\text{more} \wedge T \supset \Box_f(\text{more} \supset T)$	1, **Prop**

qed

$\vdash \quad \Box_f(\text{skip} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$ BfSkipImpAndNextImpAndSkipAndChop

Proof:

1	$\Box_f(\text{skip} \supset f) \wedge (\text{skip} \frown g) \supset ((\text{skip} \supset f) \wedge \text{skip}) \frown g$	**BfAndChopImport**
2	$(\text{skip} \supset f) \wedge \text{skip} \supset \text{skip} \wedge f$	**Prop**
3	$((\text{skip} \supset f) \wedge \text{skip}) \frown g \supset (\text{skip} \wedge f) \frown g$	2, **LeftChopImpChop**
4	$\Box_f(\text{skip} \supset f) \wedge (\text{skip} \frown g) \supset (\text{skip} \wedge f) \frown g$	1, 3, **Prop**
5	$\Box_f(\text{skip} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$	4, def. of $\bigcirc$

qed

$\vdash \quad \Box_f(\text{more} \supset f) \supset \Box_f(\text{skip} \supset f)$ BfMoreImpImpBfSkipImp

Proof:

1	$\text{more} \supset \text{skip}$	**VPTL**
2	$(\text{more} \supset f) \supset (\text{skip} \supset f)$	1, **Prop**
3	$\Box_f(\text{more} \supset f) \supset \Box_f(\text{skip} \supset f)$	2, **BfImpBfRule**

qed

$\vdash \quad \Box_f(\text{more} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$ BfMoreImpAndNextImpAndSkipAndChop

Proof:

1	$\Box_f(\text{more} \supset f) \supset \Box_f(\text{skip} \supset f)$	**BfMoreImpImpBfSkipImp**
2	$\Box_f(\text{skip} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$	**BfSkipImpAndNextImpAndSkipAndChop**
3	$\Box_f(\text{more} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$	1, 2, **Prop**

qed

$\vdash \quad \Diamond_f(\text{skip} \wedge T) \equiv \text{more} \wedge T$ DfSkipAndNLoneEqvMoreAndNLone

Proof:

1	$\text{finite} \supset \Diamond_f(\text{skip} \wedge T) \equiv (\text{more} \wedge T)$	**PITLF**
2	$\Diamond_f \Diamond_f(\text{skip} \wedge T) \equiv \Diamond_f(\text{more} \wedge T)$	1, **FiniteImpDfEqvDfRule**
3	$\Diamond_f \Diamond_f(\text{skip} \wedge T) \equiv \Diamond_f(\text{skip} \wedge T)$	**DfDfEqvDf**
4	$\Diamond_f(\text{more} \wedge T) \equiv \text{more} \wedge T$	**DfMoreAndNLoneEqvMoreAndNLone**
5	$\Diamond_f(\text{skip} \wedge T) \equiv \text{more} \wedge T$	$2 - -4$, **Prop**

qed

$\vdash \quad \Diamond(\mathsf{skip} \wedge T) \supset \Box(\mathsf{skip} \supset T)$ DfSkipAndNLoneImpBfSkipImpNLone

Proof:

1	$\Diamond(\mathsf{skip} \wedge T) \equiv \mathsf{more} \wedge T$	DfSkipAndNLoneEqvMoreAndNLone
2	$\Diamond(\mathsf{skip} \wedge \neg T) \equiv \mathsf{more} \wedge \neg T$	DfSkipAndNLoneEqvMoreAndNLone
3	$\mathsf{more} \wedge T \supset \neg(\mathsf{more} \wedge T)$	Prop
4	$\Diamond(\mathsf{skip} \wedge T) \supset \neg\Diamond(\mathsf{skip} \wedge \neg T)$	$1 - -3$, Prop
5	$\mathsf{skip} \wedge \neg T \equiv \neg(\mathsf{skip} \supset T)$	Prop
6	$\Diamond(\mathsf{skip} \wedge \neg T) \equiv \Diamond\neg(\mathsf{skip} \supset T)$	5, DfEqvDf
7	$\neg\Diamond(\mathsf{skip} \wedge \neg T) \equiv \neg\Diamond\neg(\mathsf{skip} \supset T)$	6, Prop
8	$\Diamond(\mathsf{skip} \wedge T) \supset \neg\Diamond\neg(\mathsf{skip} \supset T)$	4, 7, Prop
9	$\Diamond(\mathsf{skip} \wedge T) \supset \Box(\mathsf{skip} \supset T)$	8, def. of $\Box$

qed

$\vdash \quad (\mathsf{skip} \wedge T) \frown f \equiv T \wedge \bigcirc f$ NLoneAndSkipChopEqvNLoneAndNext

Proof for $\supset$:

1	$(\mathsf{skip} \wedge T) \frown f \supset \Diamond(\mathsf{skip} \wedge T)$	ChopImpDf
2	$\Diamond(\mathsf{skip} \wedge T) \equiv \mathsf{more} \wedge T$	DfSkipAndNLoneEqvMoreAndNLone
3	$(\mathsf{skip} \wedge T) \frown f \supset T$	1, 2, Prop
4	$(\mathsf{skip} \wedge T) \frown f \supset \mathsf{skip} \frown f$	AndChopA
5	$(\mathsf{skip} \wedge T) \frown f \supset \bigcirc f$	4, def. of $\bigcirc$
6	$(\mathsf{skip} \wedge T) \frown f \supset T \wedge \bigcirc f$	3, 5, Prop

qed

Proof for $\subset$:

1	$\bigcirc f \supset \mathsf{more}$	VPTL
2	$\mathsf{more} \wedge T \supset \Box(\mathsf{more} \supset T)$	MoreAndNLoneImpBfMoreImpNLone
3	$T \wedge \bigcirc f \supset \Box(\mathsf{more} \supset T)$	1, 2, Prop
4	$\Box(\mathsf{more} \supset T) \wedge \bigcirc f \supset (\mathsf{skip} \wedge T) \frown f$	BfMoreImpAndNextImpAndSkipAndChop
5	$T \wedge \bigcirc f \supset (\mathsf{skip} \wedge T) \frown f$	3, 4, Prop

qed

$\vdash \quad T \,\mathsf{until}\, f \equiv f \vee (T \wedge \bigcirc(T \,\mathsf{until}\, f))$ UntilEqv

Proof:

1	$\mathsf{skip} \wedge T \supset \mathsf{more}$	VPTL
2	$(\mathsf{skip} \wedge T)^{\star} \frown f \equiv f \vee ((\mathsf{skip} \wedge T) \frown ((\mathsf{skip} \wedge T)^{\star} \frown f))$	1, ImpMoreChopStarChopEqvRule
3	$T \,\mathsf{until}\, f \equiv f \vee ((\mathsf{skip} \wedge T) \frown (T \,\mathsf{until}\, f))$	2, def. of until
4	$(\mathsf{skip} \wedge T) \frown (T \,\mathsf{until}\, f) \equiv T \wedge \bigcirc(T \,\mathsf{until}\, f)$	NLoneAndSkipChopEqvNLoneAndNext
5	$T \,\mathsf{until}\, f \equiv f \vee (T \wedge \bigcirc(T \,\mathsf{until}\, f))$	$3 - -4$, Prop

qed

$\vdash \quad T \text{ until } f \supset \Diamond f$ UntilImpDiamond

Proof:

1	$(\text{skip} \wedge T)^{\star} \frown f \supset \Diamond f$	**ChopImpDiamond**
2	$T \text{ until } f \supset \Diamond f$	1, def. of until

qed

List of Theorems