

Interval Temporal Logic Proofs

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Part I

Imperative Reasoning proofs

Proofs taken from

Ben C. Moszkowski. “Imperative Reasoning in Interval Temporal Logic”, Internal Report, University of Newcastle upon Tyne.

1 A Proof System

1.1 Propositional Axioms and Inference Rules

The proof system uses a number of the propositional axioms suggested by Rosner and Pnueli but also includes our own axioms and inference rules for the operators \Box and *chop-plus*.

\vdash	All substitution instances of valid propositional <i>chop</i> -free temporal logic formulas for finite time	PTL
\vdash	$(f ; g) ; h \equiv f ; (g ; h)$	ChopAssoc
\vdash	$(f \vee f_1) ; g \supset (f ; g) \vee (f_1 ; g)$	OrChopImp
\vdash	$f ; (g \vee g_1) \supset (f ; g) \vee (f ; g_1)$	ChopOrImp
\vdash	$\text{empty} ; f \equiv f$	EmptyChop
\vdash	$f ; \text{empty} \equiv f$	ChopEmpty
\vdash	$\Box(f \supset f_1) \wedge \Box(g \supset g_1) \supset (f ; g) \supset (f_1 ; g_1)$	BiBoxChopImpChop
\vdash	$w \supset \Box w$	StatImpBi
\vdash	$\bigcirc f \supset \neg \bigcirc \neg f$	NextImpNotNextNot
\vdash	$\Diamond(\text{skip} \wedge f) \supset \Box(\text{skip} \supset f)$	DiSkiImpBiSkip
\vdash	$\Box(f \supset \textcircled{w}f) \wedge f \supset \Box f$	BoxInduct
\vdash	$f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	ChopPlusEqv
\vdash	$f \supset g, \vdash f \Rightarrow \vdash g$	MP
\vdash	$f \Rightarrow \vdash \Box f$	BoxGen
\vdash	$f \Rightarrow \vdash \Box f$	BiGen

Instead of **ChopPlusEqv** one can use

\vdash	$f^* \equiv \text{empty} \vee (f \wedge \text{more}) ; f^*$	ChopStarEqv
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$$\vdash (f \wedge f_1); g \supset f; g$$

AndChopA

Proof:

- | | |
|--|------------------|
| 1 $\vdash f \wedge f_1 \supset f$ | PTL |
| 2 $\vdash \Box(f \wedge f_1 \supset f)$ | 1, BiGen |
| 3 $\vdash \Box(f \wedge f_1 \supset f) \supset (f \wedge f_1); g \supset f; g$ | 2, BiChopImpChop |
| 4 $\vdash (f \wedge f_1); g \supset f; g$ | 2, 3, MP |

The following related theorem has a similar proof:

$$\vdash (f \wedge f_1); g \supset f_1; g$$

AndChopB

1.2 First-Order Axioms and Inference Rules

Below are axioms and inference rules for reasoning about first-order concepts. They are to be used together with the ones already introduced. See Manna and Kröger for proof systems for *chop*-free first-order temporal logic. We let v and v_1 refer to both static and state variables.

-
- | | |
|---|-----------------------------------|
| \vdash All substitution instances of valid nonmodal formulas of conventional first-order logic with arithmetic | BasicFOL |
| $\vdash \forall v \bullet f \supset f [e / v]$,
where the expression e has the same data and temporal type as the variable v and is free for v in f . | ForallElim |
| $\vdash \forall v \bullet (f \supset g) \supset (f \supset \forall v \bullet g)$,
where v doesn't occur freely in f . | ForallImpExportLeft |
| $\vdash \Box(v_1 = v_2) \supset f \equiv f [v_2 / v_1]$. | SubstAxiom |
| $\vdash w \supset \textcircled{w}$,
where w only contains static variables. | StaticWeakNext |
| $\vdash \exists v \bullet (f; g) \supset (\exists v \bullet f); g$,
where v doesn't occur freely in g . | OuterExistsChopImpLeftExistsChop |
| $\vdash \exists v \bullet (f; g) \supset f; (\exists v \bullet g)$,
where v doesn't occur freely in f . | OuterExistsChopImpRightExistsChop |
| $\vdash f \Rightarrow \vdash \forall v \bullet f$,
for any variable v . | ForallGen |
-

2 Propositional proofs

Before we prove some theorems, it is worth mentioning a few useful theorems and derived rules. They are all solely based on propositional logic and temporal logic without *chop* and we omit their proofs.

$$\begin{aligned} &\vdash f_1 \supset f_2, \dots, \vdash f_{n-1} \supset f_n \Rightarrow \vdash f_1 \supset f_n && \text{ImpChain} \\ &\vdash f_1 \equiv f_2, \dots, \vdash f_{n-1} \equiv f_n \Rightarrow \vdash f_1 \equiv f_n && \text{EqvChain} \\ &\vdash f_1, \vdash f_2, \dots, \vdash f_n \Rightarrow \vdash g, && \text{Prop} \\ &\text{where the formula } f_1 \wedge f_2 \wedge \dots \wedge f_n \supset g \\ &\text{is a substitution instance of a propositional tautology} \end{aligned}$$

3 Derived proof system for *chop*-free temporal logic

$$\vdash (\bigcirc f); g \equiv \bigcirc(f; g) \quad \text{NextChop}$$

Proof:

$$\begin{aligned} 1 &\vdash (\text{skip}; f); g \equiv \text{skip}; (f; g) && \text{ChopAssoc} \\ 2 &\vdash (\bigcirc f); g \equiv \bigcirc(f; g) && 1, \text{def. of } \bigcirc \end{aligned}$$

$$\vdash \Box(f \supset f_1) \supset (f; g) \supset (f_1; g) \quad \text{BiChopImpChop}$$

Proof:

$$\begin{aligned} 1 &\vdash g \supset g && \text{Prop} \\ 2 &\vdash \Box(g \supset g) && 1, \text{BoxGen} \\ 3 &\vdash \Box(f \supset f_1) \wedge \Box(g \supset g) \supset (f; g) \supset (f_1; g) && \text{BiBoxChopImpChop} \\ 4 &\vdash \Box(f \supset f_1) \supset (f; g) \supset (f_1; g) && 2, 3, \text{Prop} \end{aligned}$$

$$\vdash \Box(g \supset g_1) \supset (f; g) \supset (f; g_1) \quad \text{BoxChopImpChop}$$

Proof:

$$\begin{aligned} 1 &\vdash f \supset f && \text{Prop} \\ 2 &\vdash \Box(f \supset f) && 1, \text{BiGen} \\ 3 &\vdash \Box(f \supset f) \wedge \Box(g \supset g_1) \supset (f; g) \supset (f; g_1) && \text{BiBoxChopImpChop} \\ 4 &\vdash \Box(g \supset g_1) \supset (f; g) \supset (f; g_1) && 2, 3, \text{Prop} \end{aligned}$$

$$\vdash f \supset f_1 \Rightarrow \vdash f ; g \supset f_1 ; g$$

LeftChopImpChop

Proof:

1	$\vdash f \supset f_1$	given
2	$\vdash \Box(f \supset f_1)$	1, BiGen
3	$\vdash \Box(f \supset f_1) \supset f ; g \supset f_1 ; g$	BiChopImpChop
4	$\vdash f ; g \supset f_1 ; g$	2, 3, MP

$$\vdash g \supset g_1 \Rightarrow \vdash f ; g \supset f ; g_1$$

RightChopImpChop

1	$\vdash g \supset g_1$	given
2	$\vdash \Box(g \supset g_1)$	BoxGen
3	$\vdash \Box(g \supset g_1) \supset (f ; g) \supset (f ; g_1)$	BoxChopImpChop
4	$\vdash f ; g \supset f ; g_1$	2, 3, MP

Here is a derived rule that is a corollary of **RightChopImpChop**:

$$\vdash g \equiv g_1 \Rightarrow \vdash f ; g \equiv f ; g_1$$

RightChopEqvChop

We omit the proof.

$$\vdash f ; (g \vee g_1) \equiv f ; g \vee f ; g_1$$

ChopOrEqv

The proof for \subset is immediate from axiom **ChopOrImp**. Here is the proof for the converse:

1	$\vdash g \supset g \vee g_1$	Prop
2	$\vdash f ; g \supset f ; (g \vee g_1)$	1, RightChopImpChop
3	$\vdash g_1 \supset g \vee g_1$	Prop
4	$\vdash f ; g_1 \supset f ; (g \vee g_1)$	3, RightChopImpChop
5	$\vdash f ; g \vee f ; g_1 \supset f ; (g \vee g_1)$	2, 4, Prop

$$\vdash f \supset f_1 \vee f_2 \Rightarrow \vdash f ; g \supset (f_1 ; g) \vee (f_2 ; g)$$

OrChopImpRule

Proof:

- | | | |
|---|---|---------------------------|
| 1 | $\vdash f \supset f_1 \vee f_2$ | given |
| 2 | $\vdash f ; g \supset (f_1 \vee f_2) ; g$ | 1, LeftChopImpChop |
| 3 | $\vdash (f \vee f_1) ; g \equiv f_1 ; g \vee f_2 ; g$ | OrChopEqv |
| 4 | $\vdash f ; g \supset (f_1 ; g) \vee (f_2 ; g)$ | 2, 3, Prop |

$$\vdash f \equiv f \vee f_1 \Rightarrow \vdash f ; g \equiv (f_1 ; g) \vee (f_2 ; g)$$

OrChopEqvRule

Proof:

- | | | |
|---|---|---------------------------|
| 1 | $\vdash f \equiv f_1 \vee f_2$ | given |
| 2 | $\vdash f ; g \equiv (f_1 \vee f_2) ; g$ | 1, LeftChopEqvChop |
| 3 | $\vdash (f \vee f_1) ; g \equiv f_1 ; g \vee f_2 ; g$ | OrChopEqv |
| 4 | $\vdash f ; g \equiv (f_1 ; g) \vee (f_2 ; g)$ | 2, 3, EqvChain |

$$\vdash f \supset g \Rightarrow \vdash \circ f \supset \circ g$$

NextImpNext

Proof:

- | | | |
|---|--|-----------------------|
| 1 | $\vdash f \supset g$ | given |
| 2 | $\vdash \Box(f \supset g)$ | 1, BoxGen |
| 3 | $\vdash \Box(f \supset g) \supset (\text{skip} ; f) \supset (\text{skip} ; g)$ | BoxChopImpChop |
| 4 | $\vdash (\text{skip} ; f) \supset (\text{skip} ; g)$ | 2, 3, MP |
| 5 | $\vdash \circ f \supset \circ g$ | 4, def. of \circ |

$$\vdash \circ(f \supset g) \supset \circ f \supset \circ g$$

NextImpDist

Proof:

- | | | |
|---|--|----------------------------|
| 1 | $\vdash \neg(f \supset g) \equiv f \wedge \neg g$ | Prop |
| 2 | $\vdash \text{skip} ; \neg(f \supset g) \equiv \text{skip} ; (f \wedge \neg g)$ | 1, RightChopEqvChop |
| 3 | $\vdash f \supset g \vee (f \wedge \neg g)$ | Prop |
| 4 | $\vdash \text{skip} ; f \supset (\text{skip} ; g) \vee (\text{skip} ; (f \wedge \neg g))$ | 3, ChopOrImpRule |
| 5 | $\vdash \neg(\text{skip} ; (f \wedge \neg g)) \supset (\text{skip} ; f) \supset (\text{skip} ; g)$ | 4, Prop |
| 6 | $\vdash \neg(\text{skip} ; \neg(f \supset g)) \supset (\text{skip} ; f) \supset (\text{skip} ; g)$ | 2, 5, Prop |
| 7 | $\vdash \neg \circ \neg(f \supset g) \supset \circ f \supset \circ g$ | 6, def. of \circ |
| 8 | $\vdash \circ(f \supset g) \supset \neg \circ \neg(f \supset g)$ | NextImpNotNextNot |
| 9 | $\vdash \circ(f \supset g) \supset \circ f \supset \circ g$ | 7, 8, Prop |

$$\vdash f ; g \supset \diamond g$$

ChopImpDiamond

Proof:

- 1 $\vdash f \supset \text{true}$ **Prop**
 - 2 $\vdash f ; g \supset \text{true} ; g$ 1, **LeftChopImpChop**
 - 3 $\vdash f ; g \supset \diamond g$ 2, def. of \diamond
-

$\vdash f \supset \diamond f$

NowImpDiamond

Proof:

- 1 $\vdash \text{empty} ; f \equiv f$ **EmptyChop**
 - 2 $\vdash \text{empty} \supset \text{true}$ **Prop**
 - 3 $\vdash \text{empty} ; f \supset \text{true} ; f$ 2, **LeftChopImpChop**
 - 4 $\vdash f \supset \text{true} ; f$ 1, 3, **Prop**
 - 5 $\vdash f \supset \diamond f$ 4, def. of \diamond
-

$\vdash \circ \diamond f \supset \diamond f$

NextDiamondImpDiamond

Proof:

- 1 $\vdash (\text{skip} ; \text{true}) ; f \equiv \text{skip} ; (\text{true} ; f)$ **ChopAssoc**
 - 2 $\vdash (\text{skip} ; \text{true}) ; f \equiv \circ \diamond f$ 1, def. of \circ, \diamond
 - 3 $\vdash (\text{skip} ; \text{true}) ; f \supset \diamond f$ **ChopImpDiamond**
 - 4 $\vdash \circ \diamond f \supset \diamond f$ 2, 3, **Prop**
-

$\vdash \square f \supset f \wedge \textcircled{w} \square f$

BoxImpNowAndWeakNext

Proof:

- 1 $\vdash \neg f \supset \diamond \neg f$ **NowImpDiamond**
 - 2 $\vdash \neg \diamond \neg f \supset f$ 1, **Prop**
 - 3 $\vdash \square f \supset f$ 2, def. of \square
 - 4 $\vdash \circ \diamond \neg f \supset \diamond \neg f$ **NextDiamondImpDiamond**
 - 5 $\vdash \neg \neg \diamond \neg f \supset \diamond \neg f$ **Prop**
 - 6 $\vdash \circ \neg \neg \diamond \neg f \supset \circ \diamond \neg f$ 5, **NextImpNext**
 - 7 $\vdash \circ \neg \neg \diamond \neg f \supset \diamond \neg f$ 4, 6, **ImpChain**
 - 8 $\vdash \circ \neg \square f \supset \diamond \neg f$ 7, def. of \square
 - 9 $\vdash \neg \diamond \neg f \supset \neg \circ \neg \square f$ 8, **Prop**
 - 10 $\vdash \square f \supset \textcircled{w} \square f$ 9, def. of \square, \textcircled{w}
 - 11 $\vdash \square f \supset f \wedge \textcircled{w} \square f$ 3, 10, **Prop**
-

$$\vdash f \supset g \Rightarrow \Box f \supset \Box g$$

BoxImpBoxRule

Proof:

1	$\vdash f \supset g$	given
2	$\vdash \neg g \supset \neg f$	1, Prop
3	$\vdash \Box(\neg g \supset \neg f)$	2, BoxGen
4	$\vdash \Box(\neg g \supset \neg f) \supset (\text{true}; \neg g) \supset (\text{true}; \neg f)$	BoxChopImpChop
5	$\vdash \text{true}; \neg g \supset \text{true}; \neg f$	3, 4, MP
6	$\vdash \Diamond \neg g \supset \Diamond \neg f$	5, def. of \Diamond
7	$\vdash \neg \Diamond \neg f \supset \neg \Diamond \neg g$	6, Prop
8	$\vdash \Box f \supset \Box g$	7, def. of \Box

$$\vdash \Box(f \supset g) \supset \Box f \supset \Box g$$

BoxImpDist

Proof:

1	$\vdash f \supset g \supset \neg g \supset \neg f$	Prop
2	$\vdash \Box(f \supset g) \supset \Box(\neg g \supset \neg f)$	1, BoxImpBoxRule
3	$\vdash \Box(\neg g \supset \neg f) \supset (\text{true}; \neg g) \supset (\text{true}; \neg f)$	BoxChopImpChop
4	$\vdash \Box(f \supset g) \supset (\text{true}; \neg g) \supset (\text{true}; \neg f)$	2, 3, Prop
5	$\vdash \Box(f \supset g) \supset \Diamond \neg g \supset \Diamond \neg f$	4, def. of \Diamond
6	$\vdash \Box(f \supset g) \supset \neg \Diamond \neg f \supset \neg \Diamond \neg g$	5, Prop
7	$\vdash \Box(f \supset g) \supset \Box f \supset \Box g$	6, def. of \Box

$$\vdash \Diamond \text{empty}$$

DiamondEmpty

Proof:

1	$\vdash \text{true}$	Prop
2	$\vdash \text{true}; \text{empty} \equiv \text{true}$	ChopEmpty
3	$\vdash \text{true}; \text{empty}$	1, 2, Prop
4	$\vdash \Diamond \text{empty}$	3, def. of \Diamond

Here are some use derived rules for linear time temporal logic. We omit their proofs:

$$\vdash f \equiv g \Rightarrow \vdash \bigcirc f \equiv \bigcirc g$$

NextEqNext

$$\vdash f \wedge g \supset h \Rightarrow \vdash \circ f \wedge \circ g \supset \circ h$$

NextAndNextImpNextRule

$$\vdash f \wedge g \equiv h \Rightarrow \vdash \circ f \wedge \circ g \equiv \circ h$$

NextAndNextEqvNextRule

$$\vdash f \equiv g \Rightarrow \vdash \textcircled{w} f \equiv \textcircled{w} g$$

WeakNextEqvWeakNext

$$\vdash f \supset g \Rightarrow \vdash \diamond f \supset \diamond g$$

DiamondImpDiamond

$$\vdash f \equiv g \Rightarrow \vdash \diamond f \equiv \diamond g$$

DiamondEqvDiamond

$$\vdash f \supset g \Rightarrow \vdash \square f \supset \square g$$

BoxImpBox

$$\vdash f \equiv g \Rightarrow \vdash \square f \equiv \square g$$

BoxEqvBox

$$\vdash f \wedge g \supset h \Rightarrow \vdash \Box f \wedge \Box g \supset \Box h$$

BoxAndBoxImpBoxRule

$$\vdash f \wedge g \equiv h \Rightarrow \vdash \Box f \wedge \Box g \equiv \Box h$$

BoxAndBoxEqvBoxRule

$$\vdash f \supset g, \vdash \text{more} \wedge f \supset \text{O}f \Rightarrow \vdash f \supset \Box g$$

BoxIntro

$$\vdash (f \wedge \neg g) \supset \text{O}f \Rightarrow \vdash f \supset \Diamond g$$

DiamondIntro

$$\vdash f \supset \text{O}f \Rightarrow \vdash \neg f$$

NextLoop

$$\vdash f \wedge \neg g \supset \text{O}f \wedge \neg \text{O}g \Rightarrow \vdash f \supset g$$

NextContra

$$\vdash \textcircled{\text{W}}\Box f \supset f \Rightarrow \vdash f$$

WeakNextBoxInduct

$\vdash \text{empty} \supset f, \vdash \text{O}f \supset f \Rightarrow \vdash f$

EmptyNextInducta

$\vdash \text{empty} \wedge f \supset g, \vdash \text{O}(f \supset g) \wedge f \supset g \Rightarrow \vdash f \supset g$

EmptyNextInductb

$\vdash f \supset g \Rightarrow \vdash \text{fin } f \supset \text{fin } g$

FinImpFin

$\vdash f \equiv g \Rightarrow \vdash \text{fin } f \equiv \text{fin } g$

FinEqvFin

$\vdash f \wedge g \supset h \Rightarrow \vdash \text{fin } f \wedge \text{fin } g \supset \text{fin } h$

FinAndFinImpFinRule

$\vdash f \wedge g \equiv h \Rightarrow \vdash \text{fin } f \wedge \text{fin } g \equiv \text{fin } h$

FinAndFinEqvFinRule

$\vdash f \supset g \Rightarrow \vdash \text{halt } f \supset \text{halt } g$

HaltImpHalt

$$\vdash f \equiv g \Rightarrow \vdash \text{halt } f \equiv \text{halt } g$$

HaltEqvHalt

Note that **ImpChain** can be viewed as a special case of **Prop**. If desired, a deduction theorem can also be proved.

We now give proofs of some derived inference rules and theorems:

$$\vdash \Box(f \supset g) \supset \Diamond f \supset \Diamond g$$

BImpDImpDi

Proof:

- 1 $\vdash \Box(f \supset g) \supset (f ; \text{true}) \supset (g ; \text{true})$ **BiChopImpChop**
- 2 $\vdash \Box(f \supset g) \supset \Diamond f \supset \Diamond g$ 1, def. of \Diamond

$$\vdash f \supset g \Rightarrow \vdash \Diamond f \supset \Diamond g$$

DImpDi

Proof:

- 1 $\vdash f \supset g$ given
- 2 $\vdash f ; \text{true} \supset g ; \text{true}$ 1, **LeftChopImpChop**
- 3 $\vdash \Diamond f \supset \Diamond g$ 2, def. of \Diamond

Another corollary is the following:

$$\vdash f \supset g \Rightarrow \vdash \Box f \supset \Box g$$

BImpBiRule

Proof:

- 1 $\vdash f \supset g$ given
- 2 $\vdash \neg g \supset \neg f$ 1, **Prop**
- 3 $\vdash \Diamond \neg g \supset \Diamond \neg f$ 2, **DImpDi**
- 4 $\vdash \neg \Diamond \neg f \supset \neg \Diamond \neg g$ 3, **Prop**
- 5 $\vdash \Box f \supset \Box g$ 4, def. of \Box

$$\vdash f \equiv f_1 \Rightarrow \vdash f ; g \equiv f_1 ; g$$

LeftChopEqvChop

Proof:

- 1 $\vdash f \equiv f_1$ given
- 2 $\vdash f \supset f_1$ 1, **Prop**
- 3 $\vdash f ; g \supset f_1 ; g$ 2, **LeftChopImpChop**
- 4 $\vdash f_1 \supset f$ 1, **Prop**
- 5 $\vdash f_1 ; g \supset f ; g$ 4, **LeftChopImpChop**
- 6 $\vdash f ; g \equiv f_1 ; g$ 3, 5, **Prop**

Here is a corollary for the operator \diamond :

$$\vdash f \equiv g \Rightarrow \vdash \diamond f \equiv \diamond g$$

DiEqvDi

Proof:

- 1 $\vdash f \equiv f_1$ given
- 2 $\vdash f ; \text{true} \equiv g ; \text{true}$ 1, **LeftChopEqvChop**
- 3 $\vdash \diamond f \equiv \diamond g$ 2, def. of \diamond

Here is a corollary for the operator \boxplus :

$$\vdash f \equiv g \Rightarrow \vdash \boxplus f \equiv \boxplus g$$

BiEqvBi

Proof:

- 1 $\vdash f \equiv g$ given
 - 2 $\vdash \neg f \equiv \neg g$ 1, **Prop**
 - 3 $\vdash \diamond \neg f \equiv \diamond \neg g$ 2, **DiEqvDi**
 - 4 $\vdash \neg \diamond \neg f \equiv \neg \diamond \neg g$ 3, **Prop**
 - 5 $\vdash \boxplus f \equiv \boxplus g$ 4, def. of \boxplus
-

$$\vdash f ; g \supset g \Rightarrow \vdash f ; g ; h \supset g ; h$$

LeftChopChopImpChopRule

Proof:

- 1 $\vdash f ; g \supset g$ given
 - 2 $\vdash (f ; g) ; h \supset g ; h$ 1, **LeftChopImpChop**
 - 3 $\vdash (f ; g) ; h \equiv f ; g ; h$ **ChopAssoc**
 - 4 $\vdash f ; g ; h \supset g ; h$ 2, 3, **Prop**
-

$$\vdash (f \wedge f_1) ; g \equiv (f_1 \wedge f) ; g$$

AndChopCommute

Proof:

- 1 $\vdash f \wedge f_1 \equiv f_1 \wedge f$ **Prop**
 - 2 $\vdash (f \wedge f_1); g \equiv (f_1 \wedge f); g$ 1, **LeftChopEqvChop**
-

$$\vdash w \wedge f \supset f_1 \Rightarrow \vdash w \wedge (f; g) \supset (f_1; g)$$

StateAndChopImpChopRule

Proof:

- 1 $\vdash w \wedge f \supset f_1$ given
 - 2 $\vdash (w \wedge f); g \supset f_1; g$ 1, **LeftChopImpChop**
 - 3 $\vdash (w \wedge f); g \equiv w \wedge (f; g)$ **StateAndChop**
 - 4 $\vdash w \wedge f; g \supset f_1; g$ 2, 3, **Prop**
-

$$\vdash w \supset (f \equiv f_1) \Rightarrow \vdash w \supset ((f; g) \equiv (f_1; g))$$

StateImpChopEqvChop

Proof:

- 1 $\vdash w \supset f \equiv f_1$ given
 - 2 $\vdash w \wedge f \supset f_1$ 1, **Prop**
 - 3 $\vdash w \wedge (f; g) \supset (f_1; g)$ 2, **StateAndChopImpChopRule**
 - 4 $\vdash w \wedge f_1 \supset f$ 1, **Prop**
 - 5 $\vdash w \wedge (f_1; g) \supset (f; g)$ 4, **StateAndChopImpChopRule**
 - 6 $\vdash w \supset (f; g) \equiv (f_1; g)$ 3, 5, **Prop**
-

$$\vdash f \equiv w \wedge f_1 \Rightarrow \vdash (f; g) \equiv w \wedge (f_1; g)$$

ChopEqvStateAndChop

Proof:

- 1 $\vdash f \equiv w \wedge f_1$ given
 - 2 $\vdash f; g \equiv (w \wedge f_1); g$ 1, **LeftChopEqvChop**
 - 3 $\vdash (w \wedge f_1); g \equiv w \wedge (f_1; g)$ **StateAndChop**
 - 4 $\vdash (f; g) \equiv w \wedge (f_1; g)$ 2, 3, **EqvChain**
-

$$\vdash f \supset \diamond f$$

DilIntro

Proof:

1	$\vdash f ; \text{empty} \equiv f$	ChopEmpty
2	$\vdash \text{empty} \supset \text{true}$	PTL
3	$\vdash \Box(\text{empty} \supset \text{true})$	2, BoxGen
4	$\vdash \Box(\text{empty} \supset \text{true}) \supset (f ; \text{empty} \supset f ; \text{true})$	BoxChopImpChop
5	$\vdash f ; \text{empty} \supset f ; \text{true}$	3, 4, MP
6	$\vdash f ; \text{empty} \supset \Diamond f$	5, def. of \Diamond
7	$\vdash f \supset \Diamond f$	1, 6, Prop

The following is a corollary of **DilIntro**:

$$\vdash \Box f \supset f$$

BiElim

Proof:

1	$\vdash \neg f \supset \Diamond \neg f$	DilIntro
2	$\vdash (\neg f \supset \Diamond \neg f) \supset (\neg \Diamond \neg f \supset f)$	Prop
3	$\vdash \neg \Diamond \neg f \supset f$	1, 2, MP
4	$\vdash \Box f \supset f$	3, def. of \Box

The following is used in the proof of lemma **BilmpDist**:

$$\vdash \Box(\neg g \supset \neg f) \supset (\Box f) \supset (\Box g)$$

BiContraPosImpDist

Proof:

1	$\vdash \Box(\neg g \supset \neg f) \supset (\Diamond \neg g) \supset (\Diamond \neg f)$	BilmpDilmpDi
2	$\vdash \Box(\neg g \supset \neg f) \supset (\neg \Diamond \neg f) \supset (\neg \Diamond \neg g)$	1, Prop
3	$\vdash \Box(\neg g \supset \neg f) \supset (\Box f) \supset (\Box g)$	2, def. of \Box

$$\vdash \Box(f \supset g) \supset (\Box f) \supset (\Box g)$$

BilmpDist

Proof:

1	$\vdash (f \supset g) \supset (\neg g \supset \neg f)$	Prop
2	$\vdash \neg(\neg g \supset \neg f) \supset \neg(f \supset g)$	1, Prop
3	$\vdash \Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$	2, BiGen
4	$\vdash \Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g)) \supset$ $\Box(f \supset g) \supset \Box(\neg g \supset \neg f)$	BiContraPosImpDist
5	$\vdash \Box(f \supset g) \supset \Box(\neg g \supset \neg f)$	3, 4, MP
6	$\vdash \Box(\neg g \supset \neg f) \supset (\Box f) \supset (\Box g)$	BiContraPosImpDist
7	$\vdash \Box(f \supset g) \supset (\Box f) \supset (\Box g)$	5, 6, ImpChain

$$\vdash (f \vee f_1) ; g \equiv f ; g \vee f_1 ; g$$

OrChopEqv

The proof for \subset is immediate from axiom **OrChopImp**. Here is the proof for the converse:

- 1 $\vdash f \supset f \vee f_1$ **Prop**
- 2 $\vdash f ; g \supset (f \vee f_1) ; g$ **1, LeftChopImpChop**
- 3 $\vdash f_1 \supset f \vee f_1$ **Prop**
- 4 $\vdash f_1 ; g \supset (f \vee f_1) ; g$ **3, LeftChopImpChop**
- 5 $\vdash f ; g \vee f ; g_1 \supset (f \vee f_1) ; g$ **2, 4, Prop**

$$\vdash f \equiv \text{if } w \text{ then } f_1 \text{ else } f_2 \Rightarrow \vdash f ; g \equiv \text{if } w \text{ then } (f_1 ; g) \text{ else } (f_2 ; g)$$

IfChopEqvRule

Proof:

- 1 $\vdash f \equiv \text{if } w \text{ then } f_1 \text{ else } f_2$ **given**
- 2 $\vdash f \equiv (w \wedge f_1) \vee (\neg w \wedge f_2)$ **1, Prop**
- 3 $\vdash f ; g \equiv (w \wedge f_1) ; g \vee (\neg w \wedge f_2) ; g$ **2, OrChopEqvRule**
- 4 $\vdash (w \wedge f_1) ; g \equiv w \wedge (f_1 ; g)$ **StateAndChop**
- 5 $\vdash (\neg w \wedge f_2) ; g \equiv \neg w \wedge (f_2 ; g)$ **StateAndChop**
- 6 $\vdash f ; g \equiv (w \wedge f_1 ; g) \vee (\neg w \wedge f_2 ; g)$ **3, 4, 5, Prop**
- 7 $\vdash f ; g \equiv \text{if } w \text{ then } f_1 ; g \text{ else } f_2 ; g$ **6, Prop**

$$\vdash g \supset g_1 \vee g_2 \Rightarrow \vdash f ; g \supset (f ; g_1) \vee (f ; g_2)$$

ChopOrImpRule

Proof:

- 1 $\vdash g \supset g_1 \vee g_2$ **given**
- 2 $\vdash f ; g \supset f ; (g_1 \vee g_2)$ **1, RightChopImpChop**
- 3 $\vdash f ; (g_1 \vee g_2) \equiv f ; g_1 \vee f ; g_2$ **ChopOrEqv**
- 4 $\vdash f ; g \supset (f ; g_1) \vee (f ; g_2)$ **2, 3, Prop**

$$\vdash g \equiv g_1 \vee g_2 \Rightarrow \vdash f ; g \equiv (f ; g_1) \vee (f ; g_2)$$

ChopOrEqvRule

Proof:

- 1 $\vdash g \equiv g_1 \vee g_2$ **given**
- 2 $\vdash f ; g \equiv f ; (g_1 \vee g_2)$ **1, RightChopEqvChop**
- 3 $\vdash (f \vee f_1) ; g \equiv f_1 ; g \vee f_2 ; g$ **ChopOrEqv**
- 4 $\vdash f ; g \equiv (f_1 ; g) \vee (f_2 ; g)$ **2, 3, EqvChain**

$$\vdash (\text{empty} \vee f) ; g \equiv g \vee (f ; g)$$

EmptyOrChopEqv

Proof:

- 1 $\vdash (\text{empty} \vee f); g \equiv (\text{empty}; g) \vee (f; g)$ **OrChopEqv**
 - 2 $\vdash \text{empty}; g \equiv g$ **EmptyChop**
 - 3 $\vdash (\text{empty} \vee f); g \equiv g \vee (f; g)$ **1, 2, Prop**
-

$$\vdash (\text{empty} \vee \circ f); g \equiv g \vee \circ(f; g)$$

EmptyOrNextChopEqv

Proof:

- 1 $\vdash (\text{empty} \vee \circ f); g \equiv g \vee ((\circ f); g)$ **EmptyOrChopEqv**
 - 2 $\vdash (\circ f); g \equiv \circ(f; g)$ **NextChop**
 - 3 $\vdash (\text{empty} \vee \circ f); g \equiv g \vee \circ(f; g)$ **1, 2, Prop**
-

$$\vdash f \supset \text{empty} \vee f_1 \Rightarrow \vdash f; g \supset g \vee (f_1; g)$$

EmptyOrChopImpRule

Proof:

- 1 $\vdash f \supset \text{empty} \vee f_1$ **given**
- 2 $\vdash f; g \supset (\text{empty} \vee f_1); g$ **1, LeftChopImpChop**
- 3 $\vdash (\text{empty} \vee f_1); g \equiv g \vee (f_1; g)$ **EmptyOrChopEqv**
- 4 $\vdash f; g \supset g \vee (f_1; g)$ **2, 3, Prop**

Here is a related lemma:

$$\vdash f \equiv \text{empty} \vee f_1 \Rightarrow \vdash f; g \equiv g \vee (f_1; g)$$

EmptyOrChopEqvRule

Proof:

- 1 $\vdash f \equiv \text{empty} \vee f_1$ **given**
- 2 $\vdash f; g \equiv (\text{empty} \vee f); g$ **1, LeftChopEqvChop**
- 3 $\vdash (\text{empty} \vee f); g \equiv g \vee (f; g)$ **EmptyOrChopEqv**
- 4 $\vdash f; g \equiv g \vee (f; g)$ **2, 3, Prop**

The following is a useful special case of **EmptyOrChopImpRule**:

$$\vdash f \supset \text{empty} \vee \circ f_1 \Rightarrow \vdash f; g \supset g \vee \circ(f_1; g)$$

EmptyOrNextChopImpRule

Proof:

- 1 $\vdash f \supset \text{empty} \vee \circ f_1$ given
- 2 $\vdash f ; g \supset (\text{empty} \vee \circ f_1) ; g$ 1, **LeftChopImpChop**
- 3 $\vdash (\text{empty} \vee \circ f_1) ; g \equiv g \vee \circ(f_1 ; g)$ **EmptyOrNextChopEqv**
- 4 $\vdash f ; g \supset g \vee \circ(f_1 ; g)$ 2, 3, **Prop**

The following an analogous special case of **EmptyOrChopEqvRule**:

$$\vdash f \equiv \text{empty} \vee \circ f_1 \Rightarrow \vdash f ; g \equiv g \vee \circ(f_1 ; g)$$

EmptyOrNextChopEqvRule

Proof:

- 1 $\vdash f \equiv \text{empty} \vee \circ f_1$ given
- 2 $\vdash f ; g \equiv (\text{empty} \vee \circ f_1) ; g$ 1, **LeftChopEqvChop**
- 3 $\vdash (\text{empty} \vee \circ f_1) ; g \equiv g \vee \circ(f_1 ; g)$ **EmptyOrNextChopEqv**
- 4 $\vdash f ; g \equiv g \vee \circ(f_1 ; g)$ 2, 3, **Prop**

Here is a corollary of **ChopOrImpRule**:

$$\vdash g \supset \text{empty} \vee g_1 \Rightarrow \vdash f ; g \supset f \vee (f ; g_1)$$

ChopEmptyOrImpRule

Proof:

- 1 $\vdash g \supset \text{empty} \vee g_1$ given
- 2 $\vdash f ; g \supset (f ; \text{empty}) \vee (f ; g_1)$ 1, **ChopOrImpRule**
- 3 $\vdash f ; \text{empty} \equiv f$ **ChopEmpty**
- 4 $\vdash f ; g \supset f \vee (f ; g_1)$ 2, 3, **Prop**

$$\vdash \Box w ; \Box w \equiv \Box w$$

BoxStateChopBoxEqvBox

Proof for \supset :

- 1 $\vdash \Box w \equiv w \wedge (\text{empty} \vee \circ \Box w)$ **PTL**
- 2 $\vdash \Box w ; \Box w \equiv w \wedge ((\text{empty} \vee \circ \Box w) ; \Box w)$ 1, **ChopEqvStateAndChop**
- 3 $\vdash (\text{empty} \vee \circ \Box w) ; \Box w \equiv \Box w \vee \circ(\Box w ; \Box w)$ **EmptyOrNextChopEqv**
- 4 $\vdash \Box w ; \Box w \equiv w \wedge (\Box w \vee \circ(\Box w ; \Box w))$ 2, 3, **Prop**
- 5 $\vdash \neg \Box w \supset \neg w \vee \neg \circ \Box w$ **PTL**
- 6 $\vdash (\Box w ; \Box w) \wedge \neg \Box w \supset \circ(\Box w ; \Box w) \wedge \neg \circ \Box w$ 4, 5, **Prop**
- 7 $\vdash \Box w ; \Box w \supset \Box w$ 6, **NextContra**

Proof for \subset :

- 1 $\vdash \Box w \equiv w \wedge \Box w$ **PTL**
- 2 $\text{empty} ; \Box w \equiv \Box w$ **EmptyChop**
- 3 $\vdash (w \wedge \text{empty}) ; \Box w \equiv w \wedge (\text{empty} ; \Box w)$ **StateAndChop**
- 4 $\vdash \Box w \equiv (w \wedge \text{empty}) ; \Box w$ 1, 2, 3, **Prop**
- 5 $\vdash w \wedge \text{empty} \supset \Box w$ **PTL**
- 6 $\vdash (w \wedge \text{empty}) ; \Box w \supset \Box w ; \Box w$ 5, **LeftChopImpChop**
- 7 $\vdash \Box w \supset \Box w ; \Box w$ 4, 6, **Prop**

$$\vdash \neg \Box w \supset (\Box w) \rightsquigarrow \neg \Box w$$

NotBoxStatImpBoxYieldsNotBox

Proof:

- 1 $\vdash \Box w ; \Box w \equiv \Box w$ **BoxStateChopBoxEqvBox**
- 2 $\vdash \Box w \equiv \neg \neg \Box w$ **Prop**
- 3 $\vdash \Box w ; \Box w \equiv \Box w ; \neg \neg \Box w$ 2, **RightChopEqvChop**
- 4 $\vdash \neg \Box w \supset \neg(\Box w ; \neg \neg \Box w)$ 1, 3, **Prop**
- 5 $\vdash \neg \Box w \supset (\Box w) \rightsquigarrow \neg \Box w$ 4, def. of \rightsquigarrow

$$\vdash \Box w \wedge (f ; g) \equiv (\Box w \wedge f) ; (\Box w \wedge g)$$

BoxStateAndChopEqvChop

Proof for \supset :

- 1 $\vdash \Box w \equiv \Box \Box w$ **BoxStateEqvBaBoxState**
- 2 $\vdash \Box w \wedge (f ; g) \supset (\Box w \wedge f) ; (\Box w \wedge g)$ 1, **BaAndChopImport**

Proof for \subset :

- 1 $\vdash (\Box w \wedge f) ; (\Box w \wedge g) \supset (\Box w) ; (\Box w \wedge g)$ **AndChopA**
- 2 $\vdash (\Box w) ; (\Box w \wedge g) \supset (\Box w) ; (\Box w)$ **ChopAndA**
- 3 $\vdash (\Box w) ; (\Box w) \equiv \Box w$ **BoxStateChopBoxEqvBox**
- 4 $\vdash (\Box w \wedge f) ; (\Box w \wedge g) \supset f ; (\Box w \wedge g)$ **AndChopB**
- 5 $\vdash f ; (\Box w \wedge g) \supset f ; g$ **ChopAndB**
- 6 $\vdash (\Box w \wedge f) ; (\Box w \wedge g) \supset \Box w \wedge (f ; g)$ 1, 2, 3, 4, 5, **Prop**

See also the lemma **BoxStateAndCSEqvCS** for *chop-star*.

$$\vdash w \equiv \Box w$$

StateEqvBi

Proof:

- 1 $\vdash w \supset \Box w$ **StateImpBi**
- 2 $\vdash \Box w \supset w$ **BiElim**
- 3 $\vdash w \equiv \Box w$ 1, 2, **Prop**

$$\vdash \Diamond \neg f \equiv \neg \Box f$$

DiNotEqvNotBi

Proof:

- 1 $\vdash \Box f \equiv \neg \Diamond \neg f$ def. of \Box
- 2 $\vdash \Diamond \neg f \equiv \neg \Box f$ 1, **Prop**

$$\vdash \diamond f \equiv \neg \Box \neg f$$

DiEqvNotBiNot

Proof:

- 1 $\vdash \Box \neg f \equiv \neg \diamond \neg \neg f$ def. of \Box
 - 2 $\vdash \diamond \neg \neg f \equiv \neg \Box \neg f$ 1, **Prop**
 - 3 $f \equiv \neg \neg f$ **Prop**
 - 4 $\vdash \diamond f \equiv \diamond \neg \neg f$ 3, **DiEqvDi**
 - 5 $\vdash \diamond f \equiv \neg \Box \neg f$ 2, 4, **EqvChain**
-

$$\vdash \Box h \wedge f ; g \supset f ; (h \wedge g)$$

BoxAndChopImport

Proof:

- 1 $\vdash h \supset g \supset (h \wedge g)$ **Prop**
 - 2 $\vdash \Box h \supset \Box(g \supset (h \wedge g))$ 1, **BoxImpBox**
 - 3 $\vdash \Box(g \supset (h \wedge g)) \supset f ; g \supset f ; (h \wedge g)$ **BoxChopImpChop**
 - 4 $\vdash \Box h \wedge f ; g \supset f ; (h \wedge g)$ 2, 3, **Prop**
-

$$\vdash f ; g \wedge \Box h \supset f ; (g \wedge h)$$

ChopAndBoxImport

Proof:

- 1 $\vdash \Box h \wedge f ; g \supset f ; (h \wedge g)$ **BoxAndChopImport**
- 2 $\vdash f ; (h \wedge g) \equiv f ; (g \wedge h)$ **ChopAndCommute**
- 3 $\vdash \Box h \wedge f ; g \supset f ; (g \wedge h)$ 1, 2, **Prop**

The following are easily proved:

$$\vdash f ; (g \wedge g_1) \supset f ; g$$

ChopAndA

$$\vdash f ; (g \wedge g_1) \supset f ; g_1$$

ChopAndB

$$\vdash f ; (g \wedge g_1) \equiv f ; (g_1 \wedge g)$$

ChopAndCommute

$$\vdash (f \wedge g) ; (f_1 \wedge g_1) \equiv (g \wedge f) ; (g_1 \wedge f_1)$$

AndChopAndCommute

Proof:

- 1 $\vdash (f \wedge g) ; (f_1 \wedge g_1) \equiv (g \wedge f) ; (f_1 \wedge g_1)$ **AndChopCommute**
- 2 $\vdash (g \wedge f) ; (f_1 \wedge g_1) \equiv (g \wedge f) ; (g_1 \wedge f_1)$ **ChopAndCommute**
- 3 $\vdash (f \wedge g) ; (f_1 \wedge g_1) \equiv (g \wedge f) ; (g_1 \wedge f_1)$ 1, 2, **EqvChain**

$$\vdash f \supset f_1, \vdash g \supset g_1 \Rightarrow \vdash f ; g \supset f_1 ; g_1$$

ChopImpChop

Proof:

- 1 $\vdash f \supset f_1$ given
- 2 $\vdash f ; g \supset f_1 ; g$ 1, **LeftChopImpChop**
- 3 $\vdash g \supset g_1$ given
- 4 $\vdash f_1 ; g \supset f_1 ; g_1$ 3, **RightChopImpChop**
- 5 $\vdash f ; g \supset f_1 ; g_1$ 2, 4, **ImpChain**

$$\vdash f \equiv f_1, \vdash g \equiv g_1 \Rightarrow \vdash f ; g \equiv f_1 ; g_1$$

ChopEqvChop

Proof:

- 1 $\vdash f \equiv f_1$ given
- 2 $\vdash f ; g \equiv f_1 ; g$ 1, **LeftChopEqvChop**
- 3 $\vdash g \equiv g_1$ given
- 4 $\vdash f_1 ; g \equiv f_1 ; g_1$ 3, **RightChopEqvChop**
- 5 $\vdash f ; g \equiv f_1 ; g_1$ 2, 4, **EqvChain**

$$\vdash f_1 \supset g_1, \dots, \vdash f_n \supset g_n \Rightarrow \vdash (f_1 ; \dots ; f_n) \supset (g_1 ; \dots ; g_n)$$

MultChopImpChop

Proof is by induction on n .

Proof for $n = 1$:

1 $f_1 \supset g_1$ given

Proof for $n > 1$:

1 $\vdash f_1 \supset g_1$ given

2 $\vdash f_i \supset g_i, \text{ for } 2 \leq i \leq n$ given

3 $\vdash (f_2; \dots; f_n) \supset (g_2; \dots; g_n)$ 2, *induction*

4 $\vdash f_1; f_2; \dots; f_n \supset g_1; g_2; \dots; g_n$ 1, 3, **ChopImpChop**

$\vdash \Box h \supset f; g \supset f; (\Box h \wedge g)$

BoxChopImpChopBox

Proof:

1 $\vdash \Box h \supset \Box(g \supset \Box h \wedge g)$

PTL

2 $\vdash \Box(g \supset \Box h \wedge g) \supset f; g \supset f; (\Box h \wedge g)$

BoxChopImpChop

3 $\vdash \Box h \supset f; g \supset f; (\Box h \wedge g)$

1, 2, **Prop**

$\vdash \neg(f; g) \equiv f \rightsquigarrow \neg g$

NotChopEqvYieldsNot

1 $\vdash g \equiv \neg \neg g$

Prop

2 $\vdash f; g \equiv f; \neg \neg g$

1, **RightChopEqvChop**

3 $\vdash \neg(f; g) \equiv \neg(f; \neg \neg g)$

2, **Prop**

4 $\vdash \neg(f; g) \equiv \neg(f \rightsquigarrow g)$ def. of \rightsquigarrow

The following lemma **TrueChopEqvDiamond** is no longer needed since \diamond is now defined in terms of *chop*:

$\vdash \text{true}; f \equiv \diamond f$

TrueChopEqvDiamond

The proof of \supset is immediate by the following former axiom:

$\vdash \text{true}; f \supset \diamond f$

TrueChopImpDiamond

$\vdash \diamond f \supset \text{true}; f$

DiamondImpTrueChop

Proof:

1	$\vdash \diamond f \supset f \vee \bigcirc \diamond f$	PTL
2	$\vdash \text{true} \equiv \text{empty} \vee \bigcirc \text{true}$	PTL
3	$\vdash \text{true}; f \equiv (\text{empty} \vee \bigcirc \text{true}); f$	2, LeftChopEqvChop
4	$\vdash (\text{empty} \vee \bigcirc \text{true}); f \equiv \text{empty}; f \vee (\bigcirc \text{true}); f$	OrChopEqv
5	$\vdash \text{empty}; f \equiv f$	EmptyChop
6	$\vdash (\bigcirc \text{true}); f \equiv \bigcirc(\text{true}; f)$	NextChop
7	$\vdash \text{true}; f \equiv f \vee \bigcirc(\text{true}; f)$	3, 4, 5, 6, Prop
8	$\vdash \diamond f \wedge \neg(\text{true}; f) \supset \bigcirc \diamond f \wedge \neg \bigcirc(\text{true}; f)$	1, 7, Prop
9	$\vdash \diamond f \supset \text{true}; f$	8, NextContra

$\vdash \Box f \wedge (f_1; g) \supset (f \wedge f_1); g$

BiAndChopImport

Proof:

1	$\vdash f \supset (f_1 \supset f \wedge f_1)$	Prop
2	$\vdash \Box f \supset \Box(f_1 \supset f \wedge f_1)$	1, BImpBiRule
3	$\vdash \Box(f_1 \supset f \wedge f_1) \supset f; g \supset (f \wedge f_1); g$	BiChopImpChop
4	$\vdash f; g \supset (f \wedge f_1); g$	1, 3, MP

4 Further properties of Diamond-i and Box-i

$\vdash f \supset \diamond f$

ImpDi

Proof:

1	$\vdash f; \text{empty} \equiv f$	ChopEmpty
2	$\vdash \text{empty} \supset \text{true}$	Prop
3	$\vdash f; \text{empty} \supset f; \text{true}$	2, RightChopImpChop
4	$\vdash f \supset f; \text{true}$	1, 3, Prop
5	$\vdash f \supset \diamond f$	4, def. of \diamond

$\vdash \neg \diamond \text{false}$

NotDiFalse

Proof:

1	$\vdash \text{true} \supset \Box \text{true}$	StateImpBi
2	$\vdash \text{true}$	Prop
3	$\vdash \Box \text{true}$	1, 2, MP
4	$\vdash \neg \diamond \neg \text{true}$	3, def. of \Box
5	$\vdash \neg \text{true} \equiv \text{false}$	Prop
6	$\vdash \diamond \neg \text{true} \equiv \diamond \text{false}$	5, DiEqvDi
7	$\vdash \neg \diamond \text{false}$	4, 6, Prop

$$\vdash \diamond w \equiv w$$

DiState

Proof for \supset :

- | | | |
|---|--|-----------------------|
| 1 | $\vdash \neg w \supset \Box \neg w$ | StateImpBi |
| 2 | $\vdash \neg w \supset \neg \diamond \neg \neg w$ | 1, def. of \Box |
| 3 | $\vdash (\neg w \supset \neg \diamond \neg \neg w) \supset (\diamond \neg \neg w \supset w)$ | Prop |
| 4 | $\vdash \diamond \neg \neg w \supset w$ | 2, 3, MP |
| 5 | $\vdash w \supset \neg \neg w$ | Prop |
| 6 | $\vdash \diamond w \supset \diamond \neg \neg w$ | DImpDi |
| 7 | $\vdash \diamond w \supset w$ | 4, 6, ImpChain |

Proof for \subset :

- 1 $w \supset \diamond w$ **ImpDi**

Here are two important corollaries of **DiState** that are easy to prove:

$$\vdash w ; f \supset w.$$

StateChop

$$\vdash (w \wedge f) ; g \supset w$$

StateChopExportA

The following lets us move a state formula into the left side of a *chop*:

$$\vdash w \wedge (f ; g) \supset (w \wedge f) ; g$$

StateAndChopImport

Proof:

- | | | |
|---|---|------------------------|
| 1 | $\vdash w \supset \Box w$ | StateImpBi |
| 2 | $\vdash w \wedge (f ; g) \supset \Box w \wedge (f ; g)$ | 1, Prop |
| 3 | $\vdash \Box w \wedge (f ; g) \supset (w \wedge f) ; g$ | BiAndChopImport |
| 4 | $\vdash w \wedge (f ; g) \supset (w \wedge f) ; g$ | 2, 3, ImpChain |

With this proved, we can easily combine it with theorem **StateChopExportA** to deduce the following equivalence:

$$\vdash (w \wedge f) ; g \equiv w \wedge (f ; g)$$

StateAndChop

A useful corollary used in decomposing the left side of *chop*:

$$\vdash (w \wedge \text{empty}); f \equiv w \wedge f$$

StateAndEmptyChop

Proof:

- 1 $\vdash (w \wedge \text{empty}); f \equiv w \wedge \text{empty}; f$ **StateAndChop**
 - 2 $\vdash \text{empty}; f \equiv f$ **EmptyChop**
 - 3 $\vdash (w \wedge \text{empty}); f \equiv w \wedge f$ 1, 2, **Prop**
-

$$\vdash (w \wedge \circ f); g \equiv w \wedge \circ(f; g)$$

StateAndNextChop

Proof:

- 1 $\vdash (w \wedge \circ f); g \equiv w \wedge (\circ f); g$ **StateAndChop**
 - 2 $\vdash (\circ f); g \equiv \circ(f; g)$ **NextChop**
 - 3 $\vdash (w \wedge \circ f); g \equiv w \wedge \circ(f; g)$ 1, 2, **Prop**
-

$$\vdash \circ((w \wedge f); g) \equiv \circ w \wedge \circ(f; g)$$

NextStateAndChop

Proof:

- 1 $\vdash (w \wedge f); g \equiv w \wedge f; g$ **StateAndChop**
 - 2 $\vdash \circ((w \wedge f); g) \equiv \circ(w \wedge f; g)$ 1, **NextEqvNext**
 - 3 $\vdash \circ(w \wedge f; g) \equiv \circ w \wedge \circ(f; g)$ **PTL**
 - 4 $\vdash \circ((w \wedge f); g) \equiv \circ w \wedge \circ(f; g)$ 2, 3, **EqvChain**
-

$$\vdash (w \supset (f \rightsquigarrow g)) \equiv (w \wedge f) \rightsquigarrow g$$

StateYieldsEqv

Proof:

- 1 $\vdash w \wedge f; (\neg g) \equiv (w \wedge f); \neg g$ **StateAndChop**
 - 2 $\vdash (w \supset \neg(f; \neg g)) \equiv \neg((w \wedge f); \neg g)$ 1, **Prop**
-

$$\vdash w \wedge \diamond f \equiv \diamond(w \wedge f)$$

StateAndDi

Proof:

- 1 $\vdash w \wedge f ; \text{true} \equiv (w \wedge f) ; \text{true}$ **StateAndChop**
 - 2 $\vdash w \wedge \diamond f \equiv \diamond(w \wedge f)$ 1, def. of \diamond
-

$$\vdash \diamond \circ f \equiv \circ \diamond f$$

DiNext

Proof:

- 1 $\vdash (\circ f) ; \text{true} \equiv \circ(f ; \text{true})$ **NextChop**
 - 2 $\vdash \diamond \circ f \equiv \circ \diamond f$ 1, def. of \diamond
-

$$\vdash \diamond \circ w \equiv \circ w$$

DiNextState

Proof of \supset :

- 1 $\vdash \diamond \circ w \equiv \circ \diamond w$ **DiNext**
 - 2 $\vdash \diamond w \equiv w$ **DiNextState**
 - 3 $\vdash \circ \diamond w \equiv \circ w$ **NextEqvNext**
 - 4 $\vdash \diamond \circ w \equiv \circ w$ 1, 3, 4, **EqvChain**
-

$$\vdash w \supset f \Rightarrow \vdash w \supset \Box f$$

StateImpBiGen

Proof:

- 1 $\vdash w \supset f$ given
- 2 $\vdash \neg f \supset \neg w$ 1, **Prop**
- 3 $\vdash \diamond \neg f \supset \diamond \neg w$ 2, **DiImpDi**
- 4 $\vdash \diamond \neg w \equiv \neg w$ **DiState**
- 5 $\vdash \diamond \neg f \supset \neg w$ 3, 4, **Prop**
- 6 $\vdash w \supset \neg \diamond \neg f$ 5, **Prop**
- 7 $\vdash w \supset \Box f$ 6, def. of \Box

Let us now consider the following valuable theorem:

$$\vdash f ; g \wedge \neg(f ; g_1) \supset f ; (g \wedge \neg g_1)$$

ChopAndNotChopImp

Proof:

- 1 $\vdash g \supset (g \wedge \neg g_1) \vee g_1$ **Prop**
- 2 $\vdash f ; g \supset f ; (g \wedge \neg g_1) \vee f ; g_1$ 1, **LeftChopImpChop**
- 3 $\vdash f ; g \wedge \neg(f ; g_1) \supset f ; (g \wedge \neg g_1)$ 2, **Prop**

Here is a related theorem for the *yields* operator:

$$\vdash f ; g \wedge f \rightsquigarrow g_1 \supset f ; (g \wedge g_1)$$

ChopAndYieldsImp

This shows how *yields* adds information to the right of a suitable *chop* formula.

Proof:

- | | | |
|---|---|-------------------------------|
| 1 | $\vdash g \supset (g \wedge g_1) \vee \neg g_1$ | Prop |
| 2 | $\vdash f ; g \supset f ; (g \wedge g_1) \vee f ; \neg g_1$ | 1, LeftChopImpChop |
| 3 | $\vdash f ; g \wedge \neg(f ; \neg g_1) \supset f ; (g \wedge g_1)$ | 2, Prop |
| 4 | $\vdash f ; g \wedge f \rightsquigarrow g_1 \supset f ; (g \wedge g_1)$ | 3, def. of \rightsquigarrow |

Here is a corollary:

$$\vdash f ; g \wedge f \rightsquigarrow (g \supset g_1) \supset f ; g_1$$

ChopAndYieldsMP

- | | | |
|---|---|----------------------------|
| 1 | $\vdash f ; g \wedge f \rightsquigarrow (g \supset g_1) \supset f ; (g \wedge (g \supset g_1))$ | ChopAndYieldsImp |
| 2 | $\vdash g \wedge (g \supset g_1) \supset g_1$ | Prop |
| 3 | $\vdash f ; (g \wedge (g \supset g_1)) \supset f ; g_1$ | 2, RightChopImpChop |
| 4 | $\vdash f ; g \wedge f \rightsquigarrow (g \supset g_1) \supset f ; g_1$ | 1, 3, ImpChain |

$$\vdash (f \vee f_1) \rightsquigarrow g \equiv (f \rightsquigarrow g) \wedge (f_1 \rightsquigarrow g)$$

OrYieldsImp

Proof:

- | | | |
|---|--|-------------------------------|
| 1 | $\vdash (f \vee f_1) ; \neg g \equiv f ; \neg g \vee f_1 ; \neg g$ | OrChopEqv |
| 2 | $\vdash \neg((f \vee f_1) ; \neg g) \equiv \neg(f ; \neg g) \wedge \neg(f_1 ; \neg g)$ | 1, Prop |
| 3 | $\vdash (f \vee f_1) \rightsquigarrow g \equiv (f \rightsquigarrow g) \wedge (f_1 \rightsquigarrow g)$ | 2, def. of \rightsquigarrow |

$$\vdash f \supset f_1 \Rightarrow \vdash (f_1 \rightsquigarrow g) \supset (f \rightsquigarrow g)$$

LeftYieldsImpYields

Proof:

- | | | |
|---|--|-------------------------------|
| 1 | $\vdash f \supset f_1$ | given |
| 2 | $\vdash f ; \neg g \supset f_1 ; \neg g$ | 1, LeftChopImpChop |
| 3 | $\vdash \neg(f_1 ; \neg g) \supset \neg(f ; \neg g)$ | 2, Prop |
| 4 | $\vdash f_1 \rightsquigarrow g \supset f \rightsquigarrow g$ | 3, def. of \rightsquigarrow |

$$\vdash f \equiv f_1 \Rightarrow \vdash (f \rightsquigarrow g) \equiv (f_1 \rightsquigarrow g)$$

LeftYieldsEqYields

Proof:

- 1 $\vdash f \equiv f_1$ given
 - 2 $\vdash f ; \neg g \equiv f_1 ; \neg g$ 1, **LeftChopEqvChop**
 - 3 $\vdash \neg(f ; \neg g) \equiv \neg(f_1 ; \neg g)$ 2, **Prop**
 - 4 $\vdash f \rightsquigarrow g \equiv f_1 \rightsquigarrow g$ 3, def. of \rightsquigarrow
-

$$\vdash w \wedge f \supset \text{fin } w_1 \Rightarrow \vdash w \supset (f \rightsquigarrow w_1)$$

StateImpYields

Proof:

- 1 $\vdash w \wedge f \supset \text{fin } w_1$ given
 - 2 $\vdash w \supset (f \supset \text{fin } w_1)$ 1, **Prop**
 - 3 $\vdash w \supset \boxplus(f \supset \text{fin } w_1)$ 2, **StateImpBiGen**
 - 4 $\vdash \boxplus(f \supset \text{fin } w_1) \equiv f \rightsquigarrow w_1$ **BImpFinEqvYieldsState**
 - 5 $\vdash w \supset (f \rightsquigarrow w_1)$ 3, 4, **EqvChain**
-

$$\vdash w \wedge f \supset f_1 \Rightarrow \vdash w \wedge (f_1 \rightsquigarrow g) \supset (f \rightsquigarrow g)$$

StateAndYieldsImpYields

Proof:

- 1 $\vdash w \wedge f \supset f_1$ given
 - 2 $\vdash w \wedge (f ; \neg g) \supset f_1 ; \neg g$ 1, **StateAndChopImpChopRule**
 - 3 $\vdash w \wedge \neg(f_1 ; \neg g) \supset \neg(f ; \neg g)$ 2, **Prop**
 - 4 $\vdash w \wedge (f_1 \rightsquigarrow g) \supset (f \rightsquigarrow g)$ 3, def. of \rightsquigarrow
-

$$\vdash f \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$$

AndYieldsA

Proof:

- 1 $\vdash f \wedge f_1 \supset f$ **Prop**
 - 2 $\vdash f \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$ 1, **LeftYieldsImpYields**
-

$$\vdash f_1 \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$$

AndYieldsB

Proof:

- 1 $\vdash f \wedge f_1 \supset f_1$ **Prop**
 - 2 $\vdash f_1 \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$ 1, **LeftYieldsImpYields**
-

$$\vdash g \supset g_1 \Rightarrow \vdash (f \rightsquigarrow g) \supset (f \rightsquigarrow g_1)$$

RightYieldsImpYields

Proof:

- | | | |
|---|--|-------------------------------|
| 1 | $\vdash g \supset g_1$ | given |
| 2 | $\vdash \neg g_1 \supset \neg g$ | 1, Prop |
| 3 | $\vdash f ; \neg g_1 \supset f ; \neg g$ | 2, RightChopImpChop |
| 4 | $\vdash \neg(f ; \neg g) \supset \neg(f ; \neg g_1)$ | 3, Prop |
| 5 | $\vdash (f \rightsquigarrow g) \supset (f \rightsquigarrow g_1)$ | 3, def. of \rightsquigarrow |

$$\vdash g \equiv g_1 \Rightarrow \vdash (f \rightsquigarrow g) \equiv (f \rightsquigarrow g_1)$$

RightYieldsEqvYields

Proof:

- | | | |
|---|---|-------------------------------|
| 1 | $\vdash g \equiv g_1$ | given |
| 2 | $\vdash \neg g \equiv \neg g_1$ | 1, Prop |
| 3 | $\vdash f ; \neg g \equiv f ; \neg g_1$ | 2, RightChopEqvChop |
| 4 | $\vdash \neg(f ; \neg g) \equiv \neg(f ; \neg g_1)$ | 3, Prop |
| 5 | $\vdash f \rightsquigarrow g \equiv f \rightsquigarrow g_1$ | 4, def. of \rightsquigarrow |

$$\vdash \Box g \supset f \rightsquigarrow g$$

BoxImpYields

Proof:

- | | | |
|---|--|-------------------------------------|
| 1 | $\vdash f ; \neg g \supset \Diamond \neg g$ | ChopImpDiamond |
| 2 | $\vdash \neg \Diamond \neg g \supset \neg(f ; \neg g)$ | 1, Prop |
| 3 | $\vdash \Box g \supset f \rightsquigarrow g$ | 2, def. of \Box, \rightsquigarrow |

$$\vdash \Box f \equiv \text{true} \rightsquigarrow f$$

BoxEqvTrueYields

Proof:

- | | | |
|---|---|-------------------------------|
| 1 | $\vdash \text{true} ; \neg f \equiv \Diamond \neg f$ | TrueChopEqvDiamond |
| 2 | $\vdash \neg(\text{true} ; \neg f) \equiv \neg \Diamond \neg f$ | 1, Prop |
| 3 | $\vdash \Box f \equiv \neg \Diamond \neg f$ | PTL |
| 4 | $\vdash \Box f \equiv \neg(\text{true} ; \neg f)$ | 2, 3, Prop |
| 5 | $\vdash \Box f \equiv \text{true} \rightsquigarrow f$ | 4, def. of \rightsquigarrow |

$$\vdash g \Rightarrow \vdash f \rightsquigarrow g$$

YieldsGen

Proof:

- 1 $\vdash g$ given
 - 2 $\vdash \Box g$ **BoxGen**
 - 3 $\vdash \Box g \supset f \rightsquigarrow g$ **BoxImpYields**
 - 4 $\vdash f \rightsquigarrow g$ 2, 3, **MP**
-

$$\vdash (f \rightsquigarrow g) \wedge (f \rightsquigarrow g_1) \equiv f \rightsquigarrow (g \wedge g_1)$$

YieldsAndYieldsEqvYieldsAnd

Proof:

- 1 $\vdash f ; (\neg g \vee \neg g_1) \equiv (f ; \neg g) \vee (f ; \neg g_1)$ **ChopOrEqv**
 - 2 $\vdash (f ; \neg g) \vee (f ; \neg g_1) \equiv f ; (\neg g \vee \neg g_1)$ 1, **Prop**
 - 3 $\vdash \neg g \vee \neg g_1 \equiv \neg(g \wedge g_1)$ **Prop**
 - 4 $\vdash f ; (\neg g \vee \neg g_1) \equiv f ; \neg(g \wedge g_1)$ 3, **RightChopEqvChop**
 - 5 $\vdash (f ; \neg g) \vee (f ; \neg g_1) \equiv f ; \neg(g \wedge g_1)$ 2, 4, **ImpChain**
 - 6 $\vdash \neg(f ; \neg g) \wedge \neg(f ; \neg g_1) \equiv \neg(f ; \neg(g \wedge g_1))$ 5, **Prop**
 - 7 $\vdash (f \rightsquigarrow g) \wedge (f \rightsquigarrow g_1) \equiv f \rightsquigarrow (g \wedge g_1)$ 6, def. of \rightsquigarrow
-

$$\vdash (f \rightsquigarrow g) \wedge (f_1 \rightsquigarrow g_1) \supset (f \wedge f_1) \rightsquigarrow (g \wedge g_1)$$

YieldsAndYieldsImpAndYieldsAnd

Proof:

- 1 $\vdash f \rightsquigarrow g \supset (f \wedge f_1) \rightsquigarrow g$ **AndYieldsA**
 - 2 $\vdash f_1 \rightsquigarrow g_1 \supset (f \wedge f_1) \rightsquigarrow g_1$ **AndYieldsB**
 - 3 $\vdash (f \wedge f_1) \rightsquigarrow g \wedge (f \wedge f_1) \rightsquigarrow g_1 \equiv (f \wedge f_1) \rightsquigarrow (g \wedge g_1)$ **YieldsAndYieldsEqvYieldsAnd**
 - 4 $\vdash (f \rightsquigarrow g) \wedge (f_1 \rightsquigarrow g_1) \supset (f \wedge f_1) \rightsquigarrow (g \wedge g_1)$ 1, 2, 3, **Prop**
-

$$\vdash f \rightsquigarrow (g \rightsquigarrow h) \equiv (f ; g) \rightsquigarrow h$$

YieldsYieldsEqvChopYields

Proof:

- 1 $\vdash (f ; g) ; \neg h \equiv f ; (g ; \neg h)$ **ChopAssoc**
 - 2 $\vdash f ; (g ; \neg h) \equiv (f ; g) ; \neg h$ 1, **Prop**
 - 3 $\vdash g ; \neg h \equiv \neg\neg(g ; \neg h)$ **Prop**
 - 4 $\vdash f ; (g ; \neg h) \equiv f ; \neg\neg(g ; \neg h)$ 3, **LeftChopEqvChop**
 - 5 $\vdash f ; \neg\neg(g ; \neg h) \equiv (f ; g) ; \neg h$ 2, 4, **Prop**
 - 6 $\vdash f ; \neg(g \rightsquigarrow h) \equiv (f ; g) ; \neg h$ 5, def. of \rightsquigarrow
 - 7 $\vdash \neg(f ; \neg(g \rightsquigarrow h)) \equiv \neg((f ; g) ; \neg h)$ 6, **Prop**
 - 8 $\vdash f \rightsquigarrow (g \rightsquigarrow h) \equiv (f ; g) \rightsquigarrow h$ 7, def. of \rightsquigarrow
-

$\vdash \text{empty} \rightsquigarrow f \equiv f$

EmptyYields

Proof:

- 1 $\vdash \text{empty} ; \neg f \equiv \neg f$ **EmptyChop**
- 2 $\vdash \neg(\text{empty} ; \neg f) \equiv f$ 1, **Prop**
- 3 $\vdash \text{empty} \rightsquigarrow f \equiv f$ 2, def. of \rightsquigarrow

$\vdash (\bigcirc f) \rightsquigarrow g \equiv \textcircled{w}(f \rightsquigarrow g)$

NextYields

Proof:

- 1 $\vdash (\bigcirc f) ; \neg g \equiv \bigcirc(f ; \neg g)$ **NextChop**
- 2 $\vdash \neg((\bigcirc f) ; \neg g) \equiv \neg \bigcirc(f ; \neg g)$ 1, **Prop**
- 3 $\vdash (\bigcirc f) \rightsquigarrow g \equiv \neg \bigcirc(f ; \neg g)$ 2, def. of \rightsquigarrow
- 4 $\vdash \neg \bigcirc(f ; \neg g) \equiv \textcircled{w} \neg(f ; \neg g)$ **PTL**
- 5 $\vdash (\bigcirc f) \rightsquigarrow g \equiv \textcircled{w} \neg(f ; \neg g)$ 3, 4, **Prop**
- 6 $\vdash (\bigcirc f) \rightsquigarrow g \equiv \textcircled{w}(f \rightsquigarrow g)$ 5, def. of \rightsquigarrow

$\vdash \text{skip} ; f \equiv \bigcirc f$

SkipChopEqvNext

Proof:

- 1 $\vdash (\bigcirc \text{empty}) ; f \equiv \bigcirc(\text{empty} ; f)$ **NextChop**
- 2 $\vdash \text{empty} ; f \equiv f$ **EmptyChop**
- 3 $\vdash \bigcirc(\text{empty} ; f) \equiv \bigcirc f$ 2, **NextEqvNext**
- 4 $\vdash (\bigcirc \text{empty}) ; f \equiv \bigcirc f$ 1, 3, **Prop**
- 5 $\vdash \text{skip} ; f \equiv \bigcirc f$ 4, def. of skip

$\vdash \text{skip} \rightsquigarrow f \equiv \textcircled{w} f$

SkipYieldsEqvWeakNext

Proof:

- 1 $\vdash \text{skip} ; \neg f \equiv \bigcirc \neg f$ **SkipChopEqvNext**
- 2 $\vdash \neg(\text{skip} ; \neg f) \equiv \neg \bigcirc \neg f$ 1, **Prop**
- 3 $\vdash \neg \bigcirc \neg f \equiv \textcircled{w} f$ **PTL**
- 4 $\vdash \neg(\text{skip} ; \neg f) \equiv \textcircled{w} f$ 2, 3, **EqvChain**
- 5 $\vdash \text{skip} \rightsquigarrow f \equiv \textcircled{w} f$ 4, def. of \rightsquigarrow

$\vdash \bigcirc f \supset \text{skip} \rightsquigarrow f$

NextImpSkipYields

Proof:

- 1 $\vdash \bigcirc f \supset \textcircled{w}f$ **PTL**
 - 2 $\vdash \text{skip} \rightsquigarrow f \equiv \textcircled{w}f$ **SkipYieldsEqvWeakNext**
 - 3 $\vdash \bigcirc f \supset \text{skip} \rightsquigarrow f$ 1, 2, **Prop**
-

$\vdash \text{more} \equiv \text{skip} ; \text{true}$

[MoreEqvSkipChopTrue](#)

Proof:

- 1 $\vdash \text{skip} ; \text{true} \equiv \bigcirc \text{true}$ **SkipChopEqvNext**
 - 2 $\vdash \bigcirc \text{true} \equiv \text{skip} ; \text{true}$ 1, **Prop**
 - 3 $\vdash \text{more} \equiv \text{skip} ; \text{true}$ def. of more
-

$\vdash \text{more} ; f \supset \text{more}$

[MoreChopImpMore](#)

Proof:

- 1 $\vdash (\bigcirc \text{true}) ; f \equiv \bigcirc(\text{true} ; f)$ **NextChop**
 - 2 $\vdash \bigcirc(\text{true} ; f) \supset \text{more}$ **PTL**
 - 3 $\vdash (\bigcirc \text{true}) ; f \supset \text{more}$ 1, 2, **Prop**
 - 4 $\vdash \text{more} ; f \supset \text{more}$ 3, def. of more
-

$\vdash f ; \text{more} \supset \text{more}$

[ChopMoreImpMore](#)

Proof:

- 1 $\vdash f ; \text{more} \supset \diamond \text{more}$ **ChopImpDiamond**
 - 2 $\vdash \diamond \text{more} \supset \text{more}$ **PTL**
 - 3 $\vdash f ; \text{more} \supset \text{more}$ 1, 2, **ImpChain**
-

$\vdash \text{more} ; f \equiv \bigcirc \diamond f$

[MoreChopEqvNextDiamond](#)

Proof of \supset :

- 1 $\vdash \text{more} ; f \equiv (\bigcirc \text{true}) ; f$ def. of more
- 2 $\vdash (\bigcirc \text{true}) ; f \equiv \bigcirc(\text{true} ; f)$ **NextChop**
- 3 $\vdash \text{more} ; f \equiv \bigcirc(\text{true} ; f)$ 1, 2, **EqvChain**
- 4 $\vdash \text{more} ; f \equiv \bigcirc \diamond$ 3, def. of \diamond

Proof of \subset :

- 1 $\vdash \diamond f \supset \text{true} ; f$ **DiamondImpTrueChop**
- 2 $\vdash \bigcirc \diamond f \supset \bigcirc(\text{true} ; f)$ 1, **NextImpNext**
- 3 $\vdash (\bigcirc \text{true}) ; f \equiv \bigcirc(\text{true} ; f)$ **NextChop**
- 4 $\vdash \text{more} ; f \equiv \bigcirc(\text{true} ; f)$ 3, def. of \rightsquigarrow
- 5 $\vdash \bigcirc \diamond f \supset \text{more} ; f$ 2, 3, 4, **Prop**

The following are is an easy corollary:

$$\vdash \text{more} \rightsquigarrow f \equiv \textcircled{W} \square f$$

WeakNextBoxImpMoreYields

$$\vdash \neg f \equiv f \rightsquigarrow \text{more}$$

NotEqvYieldsMore

Proof:

- 1 $\vdash f ; \text{empty} \equiv f$ **ChopEmpty**
- 2 $\vdash \neg(f ; \text{empty}) \equiv \neg f$ 1, **Prop**
- 3 $\vdash \text{empty} \equiv \neg \text{more}$ def. of empty
- 4 $\vdash f ; \text{empty} \equiv f ; \neg \text{more}$ 3, **RightChopEqvChop**
- 5 $\vdash \neg(f ; \text{empty}) \equiv \neg(f ; \neg \text{more})$ 4, **Prop**
- 6 $\vdash \neg f \equiv \neg(f ; \neg \text{more})$ 2, 5, **EqvChain**
- 7 $\vdash \neg f \equiv f \rightsquigarrow \text{more}$ 6, def. of \rightsquigarrow

$$\vdash f \supset \text{more} \Rightarrow \vdash f ; g \supset \text{more}$$

LeftChopImpMoreRule

Proof:

- 1 $\vdash f \supset \text{more}$ given
- 2 $\vdash f ; g \supset \text{more} ; g$ 1, **LeftChopImpChop**
- 3 $\vdash \text{more} ; g \supset \text{more}$ **MoreChopImpMore**
- 4 $\vdash f ; g \supset \text{more}$ 2, 3, **ImpChain**

$$\vdash g \supset \text{more} \Rightarrow \vdash f ; g \supset \text{more}$$

RightChopImpMoreRule

Proof:

- 1 $\vdash g \supset \text{more}$ given
- 2 $\vdash f ; g \supset f ; \text{more}$ 1, **RightChopImpChop**
- 3 $\vdash f ; \text{more} \supset \text{more}$ **MoreChopImpMore**
- 4 $\vdash f ; g \supset \text{more}$ 2, 3, **ImpChain**

$$\vdash \neg \diamond f \equiv \Box \neg f$$

NotDiEqvBiNot

Proof:

- 1 $\vdash f \equiv \neg \neg f$ **Prop**
 - 2 $\vdash \diamond f \equiv \diamond \neg \neg f$ 1, **DiEqvDi**
 - 3 $\vdash \neg \diamond f \equiv \neg \diamond \neg \neg f$ 2, **Prop**
 - 4 $\vdash \neg \diamond f \equiv \Box \neg f$ 3, def. of \Box
-

$$\vdash f ; g \supset \diamond f$$

ChopImpDi

Proof:

- 1 $\vdash g \supset \text{true}$ **Prop**
 - 2 $\vdash f ; g \supset f ; \text{true}$ 1, **BoxChopImpChop**
 - 3 $\vdash f ; g \supset \diamond f$ 2, def. of \diamond
-

$$\vdash \text{true} \equiv \text{true} ; \text{true}$$

TrueEqvTrueChopTrue

Proof:

- 1 $\vdash \text{true} ; \text{true} \supset \text{true}$ **Prop**
 - 2 $\vdash \text{true} \supset \diamond \text{true}$ **DiIntro**
 - 3 $\vdash \text{true} \supset \text{true} ; \text{true}$ 2, def. of \diamond
 - 4 $\vdash \text{true} \equiv \text{true} ; \text{true}$ 1, 3, **Prop**
-

$$\vdash \diamond f \equiv \diamond \diamond f$$

DiEqvDiDi

Proof:

- 1 $\vdash \text{true} \equiv \text{true} ; \text{true}$ **TrueEqvTrueChopTrue**
 - 2 $\vdash f ; \text{true} \equiv f ; (\text{true} ; \text{true})$ 1, **RightChopEqvChop**
 - 3 $\vdash (f ; \text{true}) ; \text{true} \equiv f ; (\text{true} ; \text{true})$ **ChopAssoc**
 - 4 $\vdash f ; \text{true} \equiv (f ; \text{true}) ; \text{true}$ 2, 3, **Prop**
 - 5 $\vdash \diamond f \equiv \diamond \diamond f$ 4, def. of \diamond
-

$$\vdash \Box f \equiv \Box \Box f$$

BiEqvBiBi

Proof:

- 1 $\vdash \diamond \neg f \equiv \diamond \diamond \neg f$ **DiEqvDiDi**
 - 2 $\vdash \diamond \neg f \equiv \neg \Box f$ **DiNotEqvNotBi**
 - 3 $\vdash \diamond \diamond \neg f \equiv \diamond \neg \Box f$ 2, **DiEqvDi**
 - 4 $\vdash \diamond \neg f \equiv \diamond \neg \Box f$ 1, 3, **EqvChain**
 - 5 $\vdash \neg \diamond \neg f \equiv \neg \diamond \neg \Box f$ 4, **Prop**
 - 6 $\vdash \Box f \equiv \Box \Box f$ 5, def. of \Box
-

$$\vdash \diamond(f \vee g) \equiv \diamond f \vee \diamond g$$

DiOrEqv

Proof:

- 1 $\vdash (f \vee g); \text{true} \equiv f; \text{true} \vee g; \text{true}$ **OrChopEqv**
 - 2 $\vdash \diamond(f \vee g) \equiv \diamond f \vee \diamond g$ 1, def. of \diamond
-

$$\vdash \diamond(f \wedge g) \supset \diamond f$$

DiAndA

$$\vdash \diamond(f \wedge g) \supset \diamond g$$

DiAndB

Proof of theorem **DiAndA**:

- 1 $\vdash (f \wedge g); \text{true} \supset f; \text{true}$ **AndChopA**
 - 2 $\vdash \diamond(f \wedge g) \supset \diamond f$ 1, def. of \diamond
-

$$\vdash \diamond(f \wedge g) \supset \diamond f \wedge \diamond g$$

DiAndImpAnd

Proof:

- 1 $\vdash \diamond(f \wedge g) \supset \diamond f$ **DiAndA**
 - 2 $\vdash \diamond(f \wedge g) \supset \diamond g$ **DiAndB**
 - 3 $\vdash \diamond(f \wedge g) \supset \diamond f \wedge \diamond g$ 1, 2, **Prop**
-

$$\vdash \diamond \text{skip} \equiv \text{more}$$

DiSkipEqvMore

Proof:

- 1 $\vdash \text{skip} ; \text{true} \equiv \bigcirc \text{true}$ **SkipChopEqvNext, DiAndB**
 - 2 $\vdash \bigcirc \text{true} \equiv \text{more}$ **PTL**
 - 3 $\vdash \text{skip} ; \text{true} \equiv \text{more}$ 1, 2, **Prop**
 - 4 $\vdash \diamond \text{skip} \equiv \text{more}$ 3, def. of \diamond
-

$\vdash \diamond \text{more} \equiv \text{more}$

DiMoreEqvMore

Proof for \supset :

- 1 $\vdash \diamond(\bigcirc \text{skip}) \equiv \bigcirc \diamond \text{true}$ **DiNext**
- 2 $\vdash \bigcirc \diamond \text{skip} \supset \text{more}$ **PTL**
- 3 $\vdash \diamond \bigcirc \text{skip} \supset \text{more}$ 1, 2, **ImpChain**
- 4 $\vdash \diamond \text{more} \supset \text{more}$ 3, def. of \supset

Proof of \subset :

- 1 $\vdash \text{more} \supset \diamond \text{more}$ **ImpDi**
-

$\vdash f \equiv \text{if } w \text{ then } g \text{ else } h \Rightarrow \vdash \diamond f \equiv \text{if } w \text{ then } \diamond g \text{ else } \diamond h$

DiIfEqvRule

Proof:

- 1 $\vdash f \equiv \text{if } w \text{ then } g \text{ else } h$ given
 - 2 $\vdash f ; \text{true} \equiv \text{if } w \text{ then } (g ; \text{true}) \text{ else } (h ; \text{true})$ 1, **IfChopEqvRule**
 - 3 $\vdash \diamond f \equiv \text{if } w \text{ then } \diamond g \text{ else } \diamond h$ 2, def. of \diamond
-

$\vdash \diamond \text{empty}$

DiEmpty

Proof:

- 1 $\vdash \text{true}$ **PTL**
- 2 $\vdash \text{empty} ; \text{true} \equiv \text{true}$ **EmptyChop**
- 3 $\vdash \text{empty} ; \text{true}$ 1, 2, **Prop**
- 4 $\vdash \diamond \text{empty}$ 3, def. of \diamond

5 Properties of Diamond-a and Box-a

$\vdash \diamond f \equiv \diamond \diamond f$

DaEqvDtDi

Proof:

- | | | |
|---|---|---------------------------|
| 1 | $\vdash \text{true}; (f; \text{true}) \equiv \text{true}; (f; \text{true})$ | Prop |
| 2 | $\vdash \text{true}; (f; \text{true}) \equiv \text{true}; \diamond f$ | 1, def. of \diamond |
| 3 | $\vdash \text{true}; \diamond f \equiv \diamond \diamond f$ | TrueChopEqvDiamond |
| 4 | $\vdash \text{true}; f; \text{true} \equiv \diamond \diamond f$ | 2, 3, EqvChain |
| 5 | $\vdash \diamond f \equiv \diamond \diamond f$ | 4, def. of \diamond |
-

$$\vdash \diamond f \equiv \diamond \diamond f$$

DaEqvDiDt

Proof:

- | | | |
|---|---|---------------------------|
| 1 | $\vdash \text{true}; f \equiv \diamond f$ | TrueChopEqvDiamond |
| 2 | $\vdash (\text{true}; f); \text{true} \equiv (\diamond f); \text{true}$ | 1, LeftChopEqvChop |
| 3 | $\vdash (\text{true}; f); \text{true} \equiv \diamond \diamond f$ | 2, def. of \diamond |
| 4 | $\vdash (\text{true}; f); \text{true} \equiv \text{true}; f; \text{true}$ | ChopAssoc |
| 5 | $\vdash \text{true}; f; \text{true} \equiv \diamond \diamond f$ | 3, 4, Prop |
| 6 | $\vdash \diamond f \equiv \diamond \diamond f$ | 5, def. of \diamond |

Here is a corollary of theorems **DaEqvDtDi** and **DaEqvDiDt**:

$$\vdash \diamond \diamond f \equiv \diamond \diamond f$$

DtDiEqvDiDt

$$\vdash \boxplus f \equiv \boxplus \square f$$

BaEqvBiBt

Proof:

- | | | |
|---|---|-----------------------|
| 1 | $\vdash \diamond \neg f \equiv \diamond \diamond \neg f$ | DaEqvDiDt |
| 2 | $\vdash \diamond \neg f \equiv \neg \square f$ | PTL |
| 3 | $\vdash \diamond \diamond \neg f \equiv \diamond \neg \square f$ | 2, DiEqvDi |
| 4 | $\vdash \diamond \neg f \equiv \diamond \neg \square f$ | 1, 3, EqvChain |
| 5 | $\vdash \neg \diamond \neg f \equiv \neg \diamond \neg \square f$ | 4, Prop |
| 6 | $\vdash \neg \diamond \neg f \equiv \boxplus \square f$ | 5, def. of \boxplus |
| 7 | $\vdash \boxplus f \equiv \boxplus \square f$ | 6, def. of \boxplus |
-

$$\vdash \boxplus f \equiv \square \boxplus f$$

BaEqvBtBi

Proof:

- 1 $\vdash \Diamond \neg f \equiv \Diamond \Diamond \neg f$ **DaEqvDtDi**
- 2 $\vdash \Diamond \neg f \equiv \neg \Box f$ **DiNotEqvNotBi**
- 3 $\vdash \Diamond \Diamond \neg f \equiv \Diamond \neg \Box f$ 2, **DiamondEqvDiamond**
- 4 $\vdash \neg \Diamond \neg \Box f \equiv \Box \Box f$ **PTL**
- 5 $\vdash \neg \Diamond \neg f \equiv \Box \Box f$ 1, 3, 4, **Prop**
- 6 $\vdash \Box f \equiv \Box \Box f$ 5, def. of \Box

The following is a corollary of theorems **BaEqvBtBi** and **BaEqvBiBt**:

$$\vdash \Box \Box f \equiv \Box f$$

BtBiEqvBiBt

$$\vdash \Diamond \neg f \equiv \neg \Box f$$

DaNotEqvNotBa

Proof:

- 1 $\vdash \Box f \equiv \neg \Diamond \neg f$ def. of \Box
 - 2 $\vdash \Diamond \neg f \equiv \neg \Box f$ 1, **Prop**
-

$$\vdash \Diamond f \equiv \neg \Box \neg f$$

DaEqvNotBaNot

Proof:

- 1 $\vdash \Box \neg f \equiv \neg \Diamond \neg \neg f$ def. of \Box
 - 2 $\vdash \Diamond \neg \neg f \equiv \neg \Box \neg f$ 1, **Prop**
 - 3 $\vdash f \equiv \neg \neg f$ **Prop**
 - 4 $\vdash \Diamond f \equiv \Diamond \neg \neg f$ 3, **DaEqvDa**
 - 5 $\vdash \Diamond f \equiv \neg \Box \neg f$ 2, 4, **EqvChain**
-

$$\vdash \Box f \supset f$$

BaElim

Proof:

- 1 $\vdash \Box f \equiv \Box \Box f$ **BaEqvBtBi**
- 1 $\vdash \Box f \supset f$ **BiElim**
- 2 $\vdash \Box(\Box f \supset f)$ 1, **BoxGen**
- 3 $\vdash \Box(\Box f \supset f) \supset \Box \Box f \supset \Box f$ **PTL**
- 4 $\vdash \Box \Box f \supset \Box f$ 2, 3, **MP**
- 5 $\vdash \Box f \supset f$ **PTL**
- 6 $\vdash \Box f \supset f$ 1, 4, 5, **Prop**

Here is a corollary:

$$\vdash f \supset \lozenge f$$

DalIntro

Proof:

- 1 $\vdash \Box \neg f \supset \neg f$ **BaElim**
 - 2 $\vdash \neg \neg f \supset \neg \Box \neg f$ **1, Prop**
 - 3 $\vdash f \equiv \neg \neg f$ **Prop**
 - 4 $\vdash \lozenge f \equiv \neg \Box \neg f$ **DaEqvNotBaNot**
 - 5 $\vdash f \supset \lozenge f$ **2, 3, 4, Prop**
-

$$\vdash \Box f \supset \Box \Box f$$

BalmpBt

Proof:

- 1 $\vdash \Box f \equiv \Box \Box f$ **BaEqvBiBt**
- 2 $\vdash \Box \Box f \supset \Box f$ **BiElim**
- 3 $\vdash \Box f \supset \Box f$ **1, 2, MP**

Here is an easy corollary:

$$\vdash \lozenge f \supset \lozenge \lozenge f$$

DiamondImpDa

$$\vdash \Box f \supset \Box \Box f$$

BalmpBi

Proof:

- 1 $\vdash \Box f \equiv \Box \Box f$ **BaEqvBtBi**
- 2 $\vdash \Box \Box f \supset \Box f$ **PTL**
- 3 $\vdash \Box f \supset \Box f$ **1, 2, MP**

Here is an easy corollary:

$$\vdash \lozenge f \supset \lozenge \lozenge f$$

DilmpDa

$$\vdash f \Rightarrow \vdash \boxed{a} f$$

BaGen

Proof:

- | | | |
|---|--|------------------|
| 1 | $\vdash f$ | given |
| 2 | $\vdash \Box f$ | 1, BoxGen |
| 3 | $\vdash \Box \Box f$ | 2, BiGen |
| 4 | $\vdash \boxed{a} f \equiv \Box \Box f$ | BaEqvBiBt |
| 5 | $\vdash \Box \Box f \supset \boxed{a} f$ | 4, Prop |
| 6 | $\vdash \boxed{a} f$ | 3, 5, MP |

$$\vdash \boxed{a}(f \supset g) \supset \boxed{a} f \supset \boxed{a} g$$

BalmpDist

Proof:

- | | | |
|---|--|-------------------------|
| 1 | $\vdash \Box(f \supset g) \supset (\Box f \supset \Box g)$ | BImpDist |
| 2 | $\vdash \Box(\Box(f \supset g) \supset (\Box f \supset \Box g))$ | BoxGen |
| 3 | $\vdash \Box(\Box(f \supset g) \supset (\Box f \supset \Box g)) \supset$
$(\Box \Box(f \supset g) \supset (\Box \Box f \supset \Box \Box g))$ | PTL |
| 4 | $\vdash \Box \Box(f \supset g) \supset (\Box \Box f \supset \Box \Box g)$ | 2, 3, MP |
| 5 | $\vdash \boxed{a}(f \supset g) \equiv \Box \Box(f \supset g)$ | BaEqvBtBi |
| 6 | $\vdash \boxed{a} f \equiv \Box \Box f$ | BaEqvBtBi |
| 7 | $\vdash \boxed{a} g \equiv \Box \Box g$ | BaEqvBtBi |
| 8 | $\vdash \boxed{a}(f \supset g) \supset (\boxed{a} f \supset \boxed{a} g)$ | 4, 5, 6, 7, Prop |

Here are some easy corollaries:

$$\vdash \boxed{a}(f \equiv g) \supset \boxed{a} f \equiv \boxed{a} g$$

BalmpBaEqvBa

$$\vdash \Diamond(f \supset g) \supset \Diamond f \supset \Diamond g$$

DalmpDalmpDa

$$\vdash \Diamond(f \equiv g) \supset \Diamond f \equiv \Diamond g$$

DalmpDaEqvDa

$$\vdash f \supset g \Rightarrow \boxed{a} f \supset \boxed{a} g$$

BaImpBa

$$\vdash f \equiv g \Rightarrow \boxed{a} f \equiv \boxed{a} g$$

BaEqvBa

$$\vdash f \supset g \Rightarrow \diamond f \supset \diamond g$$

DaImpDa

$$\vdash f \equiv g \Rightarrow \diamond f \equiv \diamond g$$

DaEqvDa

$$\vdash \boxed{a}(f \wedge g) \equiv \boxed{a} f \wedge \boxed{a} g$$

BaAndEqv

$$\vdash \boxed{a}(f_1 \wedge \dots \wedge f_n) \equiv \boxed{a} f_1 \wedge \dots \wedge \boxed{a} f_n$$

BaAndGroupEqv

$$\vdash \diamond f \equiv \diamond \diamond f$$

DaEqvDaDa

Proof:

1	$\vdash \diamond f \equiv \diamond \diamond f$	DaEqvDtDi
2	$\vdash \diamond \diamond f \equiv \diamond \diamond \diamond f$	DiEqvDiDi
3	$\vdash \diamond \diamond \diamond f \equiv \diamond \diamond \diamond \diamond f$	2, DiamondImpDiamond
4	$\vdash \diamond \diamond \diamond \diamond f \equiv \diamond \diamond \diamond \diamond \diamond f$	PTL
5	$\vdash \diamond \diamond \diamond \diamond \diamond f \equiv \diamond \diamond \diamond \diamond \diamond \diamond f$	DtDiEqvDiDt
6	$\vdash \diamond \diamond \diamond \diamond \diamond \diamond f \equiv \diamond \diamond \diamond \diamond \diamond \diamond \diamond f$	5, DiamondEqvDiamond
7	$\vdash \diamond f \equiv \diamond \diamond \diamond \diamond \diamond f$	1, 3, 4, 6, EqvChain
8	$\vdash \diamond \diamond \diamond f \equiv \diamond \diamond \diamond \diamond \diamond f$	DaEqvDtDi
9	$\vdash \diamond \diamond f \equiv \diamond \diamond \diamond f$	1, DaEqvDa
10	$\vdash \diamond f \equiv \diamond \diamond f$	7, 8, 9, Prop

$$\vdash \Box f \equiv \Box \Box f$$

BaEqvBaBa

Proof:

1	$\vdash \diamond \neg f \equiv \diamond \diamond \neg f$	DaEqvDaDa
2	$\vdash \diamond \diamond \neg f \equiv \neg \Box \neg \diamond \neg f$	DaEqvNotBaNot
3	$\vdash \neg \diamond \diamond \neg f \equiv \Box \neg \diamond \neg f$	2, Prop
4	$\vdash \neg \diamond \neg f \equiv \Box \neg \diamond \neg f$	1, 3, Prop
5	$\vdash \Box f \equiv \Box \Box f$	4, def. of \Box

$$\vdash \Box(f \supset f_1) \supset f ; g \supset f ; g_1$$

BaLeftChopImpChop

Proof:

1	$\vdash \Box(f \supset f_1) \supset \Box \Box(f \supset f_1)$	BaImpBi
2	$\vdash \Box \Box(f \supset f_1) \supset f ; g \supset f_1 ; g$	BiChopImpChop
3	$\vdash \Box(f \supset f_1) \supset f ; g \supset f_1 ; g$	1, 2, Prop

$$\vdash \Box(g \supset g_1) \supset f ; g \supset f ; g_1$$

BaRightChopImpChop

Proof:

1	$\vdash \Box(g \supset g_1) \supset \Box \Box(g \supset g_1)$	BaImpBt
2	$\vdash \Box \Box(g \supset g_1) \supset f ; g \supset f ; g_1$	BoxChopImpChop
3	$\vdash \Box(g \supset g_1) \supset f ; g \supset f ; g_1$	1, 2, Prop

$$\vdash \Box f \wedge (g ; g_1) \supset (f \wedge g) ; (f \wedge g_1)$$

BaAndChopImport

Proof:

- | | | |
|---|---|-------------------------|
| 1 | $\vdash \Box f \supset \Box \Box f$ | BalmpBi |
| 2 | $\vdash \Box f \wedge (g ; g_1) \supset (f \wedge g) ; g_1$ | BiAndChopImport |
| 3 | $\vdash \Box f \supset \Box f$ | BalmpBt |
| 4 | $\vdash \Box f \wedge (f \wedge g) ; g_1 \supset (f \wedge g) ; (f \wedge g_1)$ | BoxAndChopImport |
| 5 | $\vdash \Box f \wedge (g ; g_1) \supset (f \wedge g) ; (f \wedge g_1)$ | 1, 2, 3, 4, Prop |
-

$$\vdash (f ; f_1) \wedge \Box g \supset (f \wedge g) ; (f_1 \wedge g)$$

ChopAndBalImport

Proof:

- | | | |
|---|---|--------------------------|
| 1 | $\vdash \Box g \wedge (f ; f_1) \supset (g \wedge f) ; (g \wedge f_1)$ | BaAndChopImport |
| 2 | $\vdash (g \wedge f) ; (g \wedge f_1) \equiv (f \wedge g) ; (f_1 \wedge g)$ | AndChopAndCommute |
| 2 | $\vdash (f ; f_1) \wedge \Box g \supset (f \wedge g) ; (f_1 \wedge g)$ | 1, 2, Prop |
-

$$\vdash \Box f \supset g ; g_1 \supset g ; (\Box f \wedge g_1)$$

BaChopImpChopBa

Proof:

- | | | |
|---|---|---------------------------|
| 1 | $\vdash \Box f \supset \Box (g_1 \supset \Box f \wedge g_1)$ | PTL |
| 2 | $\vdash \Box (g_1 \supset \Box f \wedge g_1) \supset g ; g_1 \supset g ; (\Box f \wedge g_1)$ | BaRightChopImpChop |
| 3 | $\vdash \Box f \supset g ; g_1 \supset g ; (\Box f \wedge g_1)$ | 1, 2, Prop |
-

$$\vdash \Box w \equiv \Box \Box w$$

BoxStateEqvBaBoxState

Proof:

- | | | |
|---|--|-------------------------|
| 1 | $\vdash w \equiv \Box w$ | StateEqvBi |
| 2 | $\vdash \Box w \equiv \Box \Box w$ | 1, BoxEqvBox |
| 3 | $\vdash \Box \Box w \equiv \Box \Box w$ | BtBiEqvBiBt |
| 4 | $\vdash \Box w \equiv \Box \Box w$ | PTL |
| 5 | $\vdash \Box \Box w \equiv \Box \Box \Box w$ | 4, BiEqvBi |
| 6 | $\vdash \Box \Box w \equiv \Box \Box \Box w$ | BaEqvBiBt |
| 7 | $\vdash \Box w \equiv \Box \Box w$ | 2, 3, 5, 6, Prop |
-

$$\vdash \Diamond \neg \Box f \supset \neg \Box f$$

DiNotBalmpNotBa

Proof:

- 1 $\vdash \Box f \equiv \Box \Box f$ **BaEqvBaBa**
- 2 $\vdash \Box \Box f \supset \Box \Box f$ **BaImpBi**
- 3 $\vdash \Box f \supset \Box \Box f$ 1, 2, **Prop**
- 4 $\vdash \Box f \supset \neg \Diamond \neg \Box f$ 3, def. of \Box
- 5 $\vdash \Diamond \neg \Box f \supset \neg \Box f$ 4, **Prop**

$$\vdash (\neg \Box f); g \supset \neg \Box f$$

NotBaChopImpNotBa

Proof:

- 1 $\vdash (\neg \Box f); g \supset \Diamond \neg \Box f$ **ChopImpDi**
- 2 $\vdash \Diamond \neg \Box f \supset \neg \Box f$ **DiNotBaImpNotBa**
- 3 $\vdash (\neg \Box f); g \supset \neg \Box f$ 1, 2, **ImpChain**

6 Properties of Fin

$$\vdash f \wedge \text{fin } w \equiv f; (w \wedge \text{empty})$$

AndFinEqvChopStateAndEmpty

Proof for \supset :

- 1 $\vdash f; \text{empty} \equiv f$ **ChopEmpty**
- 2 $\vdash \text{empty} \supset (w \wedge \text{empty}) \vee (\neg w \wedge \text{empty})$ **Prop**
- 3 $\vdash f; \text{empty} \supset f; (w \wedge \text{empty}) \vee f; (\neg w \wedge \text{empty})$ 2, **RightChopImpChop**
- 4 $\vdash f; (\neg w \wedge \text{empty}) \supset \Diamond (\neg w \wedge \text{empty})$ **ChopImpDiamond**
- 5 $\vdash \Diamond (\neg w \wedge \text{empty}) \supset \neg \text{fin } w$ **PTL**
- 6 $\vdash f \supset f; (w \wedge \text{empty}) \vee \neg \text{fin } w$ 1, 3, 4, 5, **Prop**
- 7 $\vdash f \wedge \text{fin } w \supset f; (w \wedge \text{empty})$ 6, **Prop**

Proof for \subset :

- 1 $\vdash f; (w \wedge \text{empty}) \supset f; \text{empty}$ **ChopAndB**
- 2 $\vdash f; \text{empty} \equiv f$ **ChopEmpty**
- 3 $\vdash f; (w \wedge \text{empty}) \supset \Diamond (w \wedge \text{empty})$ **ChopImpDiamond**
- 4 $\vdash \Diamond (w \wedge \text{empty}) \supset \text{fin } w$ **PTL**
- 5 $\vdash f; (w \wedge \text{empty}) \supset f \wedge \text{fin } w$ 1, 2, 3, 4, **Prop**

The following is a lemma used in the proof of theorem **FinChopEqvDiamond**.

$$\vdash (\text{fin } w); f \equiv (w \wedge f) \vee \text{O}((\text{fin } w); f)$$

FinChopEqvOr

Proof:

1	$\vdash \text{fin } w \equiv (w \wedge \text{empty}) \vee \bigcirc \text{fin } w$	PTL
2	$\vdash (\text{fin } w); f \equiv ((w \wedge \text{empty}) \vee \bigcirc \text{fin } w); f$	1, LeftChopEqvChop
3	$\vdash ((w \wedge \text{empty}) \vee \bigcirc \text{fin } w); f \equiv (w \wedge \text{empty}); f \vee (\bigcirc \text{fin } w); f$	OrChopEqv
4	$\vdash (w \wedge \text{empty}); f \equiv w \wedge f$	StateAndEmptyChop
5	$\vdash (\bigcirc \text{fin } w); f \equiv \bigcirc((\text{fin } w); f)$	NextChop
6	$\vdash (\text{fin } w); f \equiv (w \wedge f) \vee \bigcirc((\text{fin } w); f)$	2, 4, 5, Prop

$\vdash (\text{fin } w); f \equiv \diamond(w \wedge f)$ **FinChopEqvDiamond**

Proof for \supset :

1	$\vdash (\text{fin } w); f \equiv (w \wedge f) \vee \bigcirc((\text{fin } w); f)$	FinChopEqvOr
2	$\vdash \diamond(w \wedge f) \equiv (w \wedge f) \vee \bigcirc \diamond(w \wedge f)$	PTL
3	$\vdash (\text{fin } w); f \wedge \neg \diamond(w \wedge f) \equiv \bigcirc((\text{fin } w); f) \wedge \neg \bigcirc \diamond(w \wedge f)$	1, 2, Prop
4	$\vdash (\text{fin } w); f \supset \diamond(w \wedge f)$	3, NextContra

Proof of \subset :

1	$\vdash (\text{fin } w); f \equiv (w \wedge f) \vee \bigcirc((\text{fin } w); f)$	FinChopEqvOr
2	$\vdash \diamond(w \wedge f) \equiv (w \wedge f) \vee \bigcirc \diamond(w \wedge f)$	PTL
3	$\vdash \diamond(w \wedge f) \wedge \neg((\text{fin } w); f) \equiv \bigcirc \diamond(w \wedge f) \wedge \neg \bigcirc((\text{fin } w); f)$	1, 2, Prop
4	$\vdash \diamond(w \wedge f) \supset (\text{fin } w); f$	3, NextContra

The following important theorem demonstrates how to pass information from the final state of a subinterval to the starting state of a subinterval immediately following it.

$\vdash (\text{fin } w) \rightsquigarrow w$ **FinYields**

Proof:

1	$\vdash (\text{fin } w); \neg w \equiv \diamond(w \wedge \neg w)$	FinChopEqvDiamond
2	$\vdash \neg \diamond(w \wedge \neg w)$	PTL
3	$\vdash \neg((\text{fin } w); \neg w)$	1, 2, Prop
4	$\vdash (\text{fin } w) \rightsquigarrow w$	3, def. of \rightsquigarrow

Here is a related theorem whose proof uses **FinYields**:

$\vdash (f \wedge \text{fin } w); g \equiv f; (w \wedge g)$ **AndFinChopEqvStateAndChop**

Proof for \supset :

1	$\vdash (\text{fin } w) \rightsquigarrow w$	FinYields
2	$\vdash f \wedge \text{fin } w \supset \text{fin } w$	Prop
3	$\vdash (\text{fin } w) \rightsquigarrow w \supset (f \wedge \text{fin } w) \rightsquigarrow w$	2, LeftYieldsImpYields
4	$\vdash (f \wedge \text{fin } w) \rightsquigarrow w$	1, 3, MP
5	$\vdash (f \wedge \text{fin } w); g \wedge (f \wedge \text{fin } w) \rightsquigarrow w \supset (f \wedge \text{fin } w); (g \wedge w)$	ChopAndYieldsImp
6	$\vdash (f \wedge \text{fin } w); g \supset (f \wedge \text{fin } w); (g \wedge w)$	4, 5, Prop
7	$\vdash (f \wedge \text{fin } w); (g \wedge w) \supset f; (g \wedge w)$	AndChopB
8	$\vdash g \wedge w \supset w \wedge g$	Prop
9	$\vdash f; (g \wedge w) \supset f; (w \wedge g)$	8, LeftChopImpChop
10	$\vdash (f \wedge \text{fin } w); g \supset f; (w \wedge g)$	6, 7, 9, ImpChain

Prove of \subset :

1	$\vdash f \supset (f \wedge \text{fin } w) \vee \text{fin } \neg w$	PTL
2	$\vdash f; (w \wedge g) \supset ((f \wedge \text{fin } w) \vee \text{fin } \neg w); (w \wedge g)$	1, LeftChopImpChop
3	$\vdash ((f \wedge \text{fin } w) \vee \text{fin } \neg w); (w \wedge g) \equiv$ $(f \wedge \text{fin } w); (w \wedge g) \vee (\text{fin } \neg w); (w \wedge g)$	OrChopEqv
4	$\vdash (\text{fin } \neg w); (w \wedge g) \supset \diamond(\neg w \wedge (w \wedge g))$	FinChopEqvDiamond
5	$\vdash \neg \diamond(\neg w \wedge (w \wedge g))$	PTL
6	$\vdash f; (w \wedge g) \supset (f \wedge \text{fin } w); (w \wedge g)$	2, 3, 4, 5, Prop
7	$\vdash (f \wedge \text{fin } w); (w \wedge g) \supset (f \wedge \text{fin } w); g$	ChopAndB
8	$\vdash f; (w \wedge g) \supset (f \wedge \text{fin } w); g$	6, 7, ImpChain

$$\vdash \diamond(f \wedge \text{fin } w) \equiv f; w$$

[DiAndFinEqvChopState](#)

Proof:

1	$\vdash (f \wedge \text{fin } w); \text{true} \equiv f; (w \wedge \text{true})$	AndFinChopEqvStateAndChop
2	$\vdash w \wedge \text{true} \equiv w$	Prop
3	$\vdash f; (w \wedge \text{true}) \equiv f; w$	2, RightChopEqvChop
4	$\vdash (f \wedge \text{fin } w); \text{true} \equiv f; w$	1, 3, EqvChain
5	$\vdash \diamond(f \wedge \text{fin } w) \equiv f; w$	4, def. of \diamond

Here is a corollary of **DiAndFinEqvChopState**:

$$\vdash \boxplus(f \supset \text{fin } w) \equiv f \rightsquigarrow w$$

[BilmpFinEqvYieldsState](#)

Proof:

1	$\vdash \diamond(f \wedge \text{fin } \neg w) \equiv f; \neg w$	DiAndFinEqvChopState
2	$\vdash f \wedge \text{fin } \neg w \equiv f \wedge \neg \text{fin } w$	PTL
3	$\vdash f \wedge \neg \text{fin } w \equiv \neg(f \supset \text{fin } w)$	Prop
4	$\vdash f \wedge \text{fin } \neg w \equiv \neg(f \supset \text{fin } w)$	2, 3, EqvChain
5	$\vdash \diamond(f \wedge \text{fin } \neg w) \equiv \diamond \neg(f \supset \text{fin } w)$	4, DiEqvDi
6	$\vdash \diamond \neg(f \supset \text{fin } w) \equiv \neg \boxplus(f \supset \text{fin } w)$	DiNotEqvNotBi
7	$\vdash \neg \boxplus(f \supset \text{fin } w) \equiv f; \neg w$	1, 5, 6, Prop
8	$\vdash \boxplus(f \supset \text{fin } w) \equiv \neg(f; \neg w)$	7, Prop
9	$\vdash \boxplus(f \supset \text{fin } w) \equiv f \rightsquigarrow w$	8, def. of \rightsquigarrow

$\vdash f ; \text{fin } w \supset \text{fin } w$

ChopFinImpFin

Proof:

- 1 $\vdash f ; \text{fin } w \supset \diamond \text{fin } w$ **ChopImpDiamond**
- 2 $\vdash \diamond \text{fin } w \supset \text{fin } w$ **PTL**
- 3 $\vdash f ; \text{fin } w \supset \text{fin } w$ 1, 2, **ImpChain**

$\vdash \text{fin } w \supset f \rightsquigarrow \text{fin } w$

FinImpYieldsFin

Proof:

- 1 $\vdash f ; \text{fin } \neg w \supset \text{fin } \neg w$ **ChopFinImpFin**
- 2 $\vdash \text{fin } \neg w \equiv \neg \text{fin } w$ **PTL**
- 3 $\vdash f ; \text{fin } \neg w \equiv f ; \neg \text{fin } w$ 2, **RightChopEqvChop**
- 4 $\vdash f ; \neg \text{fin } w \supset \neg \text{fin } w$ 1, 2, 3, **Prop**
- 5 $\vdash \text{fin } w \supset \neg(f ; \neg \text{fin } w)$ 4, **Prop**
- 6 $\vdash \text{fin } w \supset f \rightsquigarrow \text{fin } w$ 5, def. of \rightsquigarrow

$\vdash (f ; g) \wedge \text{fin } w \equiv f ; (g \wedge \text{fin } w)$

ChopAndFin

Proof for \supset :

- 1 $\vdash \text{fin } w \supset \text{true} \rightsquigarrow \text{fin } w$ **FinImpYieldsFin**
- 2 $\vdash (f ; g) \wedge \text{fin } w \supset (f ; g) \wedge \text{true} \rightsquigarrow \text{fin } w$ 1, **Prop**
- 3 $\vdash (f ; g) \wedge \text{true} \rightsquigarrow \text{fin } w \supset f ; (g \wedge \text{fin } w)$ **ChopAndYieldsImp**
- 4 $\vdash (f ; g) \wedge \text{fin } w \supset f ; (g \wedge \text{fin } w)$ 2, 3, **ImpChain**

Proof for \subset :

- 1 $\vdash f ; (g \wedge \text{fin } w) \supset f ; g$ **ChopAndA**
- 2 $\vdash f ; (g \wedge \text{fin } w) \supset f ; \text{fin } w$ **ChopAndB**
- 3 $\vdash f ; \text{fin } w \supset \diamond \text{fin } w$ **ChopImpDiamond**
- 4 $\vdash \diamond \text{fin } w \supset \text{fin } w$ **PTL**
- 5 $\vdash f ; (g \wedge \text{fin } w) \supset (f ; g) \wedge \text{fin } w$ 1, 2, 3, 4, **Prop**

Here is a corollary used in some proofs by contradiction:

$\vdash f ; g \wedge \neg \text{fin } w \equiv f ; (g \wedge \neg \text{fin } w)$

ChopAndNotFin

- 1 $\vdash f ; g \wedge \text{fin } \neg w \equiv f ; (g \wedge \text{fin } \neg w)$ **ChopAndFin**
 - 2 $\vdash \text{fin } \neg w \equiv \neg \text{fin } w$ **PTL**
 - 3 $\vdash g \wedge \text{fin } \neg w \equiv g \wedge \neg \text{fin } w$ 2, **Prop**
 - 4 $\vdash f ; (g \wedge \text{fin } \neg w) \equiv f ; (g \wedge \neg \text{fin } w)$ 3, **LeftChopEqvChop**
 - 5 $\vdash f ; g \wedge \neg \text{fin } w \equiv f ; (g \wedge \neg \text{fin } w)$ 1, 2, 4, **Prop**
-

$$\vdash (w \supset \text{fin } w_1) ; (w_1 \supset \text{fin } w_2) \supset (w \supset \text{fin } w_2)$$

FinChopChain

Proof:

- 1 $\vdash w \wedge (w \supset \text{fin } w_1) ; (w_1 \supset \text{fin } w_2) \supset$
 $(w \wedge (w \supset \text{fin } w_1)) ; (w_1 \supset \text{fin } w_2)$ **StateAndChopImport**
 - 2 $\vdash w \wedge (w \supset \text{fin } w_1) \supset \text{fin } w_1$ **Prop**
 - 3 $\vdash (w \wedge (w \supset \text{fin } w_1)) ; (w_1 \supset \text{fin } w_2) \supset$
 $(\text{fin } w_1) ; (w_1 \supset \text{fin } w_2)$ 2, **LeftChopImpChop**
 - 4 $\vdash (\text{fin } w_1) ; (w_1 \supset \text{fin } w_2) \equiv \diamond(w_1 \wedge (w_1 \wedge \text{fin } w_2))$ **FinChopEqvDiamond**
 - 5 $\vdash \diamond(w_1 \wedge (w_1 \wedge \text{fin } w_2)) \supset \text{fin } w_2$ **PTL**
 - 6 $\vdash w \wedge (w \supset \text{fin } w_1) ; (w_1 \supset \text{fin } w_2) \supset \text{fin } w_2$ 1, 3, 4, 5, **Prop**
 - 7 $\vdash (w \supset \text{fin } w_1) ; (w_1 \supset \text{fin } w_2) \supset (w \supset \text{fin } w_2)$ 6, **Prop**
-

$$\vdash w \wedge f \supset \text{fin } w_1$$

$$\vdash w_1 \wedge f_1 \supset \text{fin } w_2$$

$$\Rightarrow \vdash w \wedge (f ; f_1) \supset \text{fin } w_2$$

ChopRule

Proof:

- 1 $\vdash w \wedge (f ; f_1) \supset (w \wedge f) ; f_1$ **StateAndChopImport**
 - 2 $\vdash w \wedge f \supset \text{fin } w_1$ given
 - 3 $\vdash (w \wedge f) ; f_1 \supset (\text{fin } w_1) ; f_1$ 2, **LeftChopImpChop**
 - 4 $\vdash (\text{fin } w_1) ; f_1 \equiv \diamond(w_1 \wedge f_1)$ **FinChopEqvDiamond**
 - 5 $\vdash w_1 \wedge f_1 \supset \text{fin } w_2$ given
 - 6 $\vdash \diamond(w_1 \wedge f_1) \supset \diamond \text{fin } w_2$ 5, **DiamondImpDiamond**
 - 7 $\vdash \diamond \text{fin } w_2 \supset \text{fin } w_2$ **PTL**
 - 8 $\vdash w \wedge (f ; f_1) \supset \text{fin } w_2$ 1, 3, 4, 6, 7, **Prop**
-

$$\vdash w \wedge f \supset f_1 \wedge \text{fin } w_1$$

$$\vdash w_1 \wedge g \supset g_1$$

$$\Rightarrow \vdash w \wedge (f ; g) \supset (f_1 ; g_1)$$

ChopRep

Proof:

1	$\vdash w \wedge f \supset f_1 \wedge \text{fin } w_1$	given
2	$\vdash w \wedge (f ; g) \supset (f_1 \wedge \text{fin } w) ; g$	1, StateAndChopImpChopRule
3	$\vdash (f_1 \wedge \text{fin } w_1) ; g \equiv f_1 ; (w_1 \wedge g)$	AndFinChopEqvStateAndChop
4	$\vdash w_1 \wedge g \supset g_1$	given
5	$\vdash f_1 ; (w_1 \wedge g) \supset f_1 ; g_1$	4, RightChopImpChop
6	$\vdash w \wedge (f ; g) \supset f_1 ; g_1$	2, 3, 5, Prop

$\vdash w \wedge f \supset f_1 \wedge \text{fin } w_1$	ChopRepAndFin
$\vdash w_1 \wedge g \supset g_1 \wedge \text{fin } w_2$	
$\Rightarrow \vdash w \wedge (f ; g) \supset (f_1 ; g_1) \wedge \text{fin } w_2$	

Proof:

1	$\vdash w \wedge f \supset f_1 \wedge \text{fin } w_1$	given
2	$\vdash w_1 \wedge g \supset g_1 \wedge \text{fin } w_2$	given
3	$\vdash w \wedge (f ; g) \supset f_1 ; (g_1 \wedge \text{fin } w_2)$	1, 2, ChopRep
4	$\vdash f_1 ; (g_1 \wedge \text{fin } w_2) \supset f_1 ; g_1$	ChopAndA
5	$\vdash f_1 ; (g_1 \wedge \text{fin } w_2) \supset f_1 ; \text{fin } w_2$	ChopAndB
6	$\vdash f_1 ; \text{fin } w_2 \supset \text{fin } w_2$	ChopFinImpFin
7	$\vdash w \wedge (f ; g) \supset (f_1 ; g_1) \wedge \text{fin } w_2$	3, 4, 5, 6, Prop

$\vdash f \supset \text{more} ; f \Rightarrow \neg f$	MoreChopLoop
---	------------------------------

Proof:

1	$\vdash f \supset \text{more} ; f$	given
2	$\vdash \text{more} ; f \equiv \bigcirc \diamond f$	MoreChopEqvNextDiamond
3	$\vdash f \supset \bigcirc \diamond f$	2, Prop
4	$\vdash \Box(f \supset \bigcirc \diamond f)$	BoxGen
5	$\vdash \Box(f \supset \bigcirc \diamond f) \supset \neg f$	PTL
6	$\vdash \neg f$	4, 5, MP

Here is a corollary:

$\vdash f \wedge \neg g \supset (\text{more} ; (f \wedge \neg g)) \Rightarrow \vdash f \supset g$	MoreChopContra
---	--------------------------------

Proof:

1	$\vdash f \wedge \neg g \supset (\text{more} ; (f \wedge \neg g))$	given
2	$\vdash \neg(f \wedge \neg g)$	1, MoreChopLoop
3	$\vdash f \supset g$	2, Prop

Here is a variant of lemma **MoreChopLoop** that is useful in proofs:

$$\vdash f \supset g; f, \vdash g \supset \text{more} \Rightarrow \neg f$$

ChopLoop

Proof:

- 1 $\vdash f \supset g; f$ given
- 2 $\vdash g \supset \text{more}$ given
- 3 $\vdash g; f \supset \text{more}; f$ 2, **LeftChopImpChop**
- 4 $\vdash f \supset \text{more}; f$ 1, 3, **ImpChain**
- 5 $\vdash \neg f$ 4, **MoreChopLoop**

Here is a variant of lemma **MoreChopContra** that is useful in proofs:

$$\vdash f \wedge \neg g \supset h; f \wedge \neg(h; g), \vdash h \supset \text{more} \Rightarrow \vdash f \supset g$$

ChopContra

Proof:

- 1 $\vdash f \wedge \neg g \supset h; f \wedge \neg(h; g)$ given
- 2 $\vdash h \supset \text{more}$ given
- 3 $\vdash h; f \wedge \neg(h; g) \supset h; (f \wedge \neg g)$ **ChopAndNotChopImp**
- 4 $\vdash h; (f \wedge \neg g) \supset \text{more}; (f \wedge \neg g)$ 2, **LeftChopImpChop**
- 5 $\vdash f \wedge \neg g \supset \text{more}; (f \wedge \neg g)$ 1, 3, 4, **ImpChain**
- 6 $\vdash f \supset g$ 5, **MoreChopContra**

7 Properties of *chop-plus*

$$\vdash f \supset f^+$$

ImpChopPlus

Proof:

- 1 $\vdash f^+ \equiv f \vee (f \wedge \text{more}); f^+$ **ChopPlusEqv**
- 2 $\vdash f \supset f^+$ 1, **Prop**

$$\vdash f; f^+ \supset f^+$$

ChopChopPlusImpChopPlus

Proof:

1	$\vdash \text{empty} \vee \text{more}$	PTL
2	$\vdash f \supset \text{empty} \vee (f \wedge \text{more})$	1, Prop
3	$\vdash f ; f^+ \supset f^+ \vee (f \wedge \text{more}) ; f^+$	2, EmptyOrChopImpRule
4	$\vdash f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	ChopPlusEqv
5	$\vdash (f \wedge \text{more}) ; f^+ \supset f^+$	4, Prop
6	$\vdash f ; f^+ \supset f^+$	3, 5, Prop

$$\vdash f^+ \equiv f \vee (f ; f^+)$$

[ChopPlusEqvOrChopChopPlus](#)

Proof for \supset :

1	$\vdash f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	ChopPlusEqv
2	$\vdash (f \wedge \text{more}) ; f^+ \supset f ; f^+$	AndChopA
3	$\vdash f^+ \supset f \vee f ; f^+$	1, 2, Prop

Proof for \subset :

1	$\vdash f \supset f^+$	ImpChopPlus
2	$\vdash \text{empty} \vee \text{more}$	PTL
3	$\vdash f \supset \text{empty} \vee (f \wedge \text{more})$	2, Prop
4	$\vdash f ; f^+ \supset f^+ \vee (f \wedge \text{more}) ; f^+$	3, EmptyOrChopImpRule
5	$\vdash f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	ChopPlusEqv
6	$\vdash (f \wedge \text{more}) ; f^+ \supset f^+$	5, Prop
7	$\vdash f \vee (f ; f^+) \supset f^+$	1, 6, Prop

$$\vdash f \wedge \neg g \supset (g \wedge \text{more}) ; f \Rightarrow \vdash f \supset g^+$$

[ChopPlusIntro](#)

Proof:

1	$\vdash f \wedge \neg g \supset (g \wedge \text{more}) ; f$	given
2	$\vdash g^+ \equiv g \vee (g \wedge \text{more}) ; g^+$	ChopPlusEqv
3	$\vdash f \wedge \neg(g^+) \supset (g \wedge \text{more}) ; f \wedge \neg((g \wedge \text{more}) ; g^+)$	1, 2, Prop
4	$\vdash g \wedge \text{more} \supset \text{more}$	Prop
5	$\vdash f \supset g^+$	3, 4, ChopContra

$$\vdash f \supset g, \vdash (f \wedge \text{more}) ; g \supset g \Rightarrow \vdash f^+ \supset g$$

[ChopPlusElim](#)

Proof:

1	$\vdash f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	ChopPlusEqv
2	$\vdash f \supset g$	given
3	$\vdash (f \wedge \text{more}) ; g \supset g$	given
4	$\vdash f^+ \wedge \neg g \supset (f \wedge \text{more}) ; f^+ \wedge \neg((f \wedge \text{more}) ; g)$	1, 2, 3, Prop
5	$\vdash f \wedge \text{more} \supset \text{more}$	Prop
6	$\vdash f^+ \supset g$	4, 5, ChopContra

$$\vdash f \supset g, \vdash f ; g \supset g \Rightarrow \vdash f^+ \supset g$$

ChopPlusElimWithoutMore

Proof:

1	$\vdash f \supset g$	given
2	$\vdash f ; g \supset g$	given
3	$\vdash (f \wedge \text{more}) ; g \supset f ; g$	AndChopA
4	$\vdash (f \wedge \text{more}) ; g \supset g$	2, 3, ImpChain
5	$\vdash f^+ \supset g$	1, 4, ChopPlusElim

$$\vdash f \supset g \Rightarrow \vdash f^+ \supset g^+$$

ChopPlusImpChopPlus

Proof:

1	$\vdash f \supset g$	given
2	$\vdash f^+ \equiv f \vee (f \wedge \text{more}) ; f^+$	ChopPlusEqv
3	$\vdash g^+ \equiv g \vee (g \wedge \text{more}) ; g^+$	ChopPlusEqv
4	$\vdash f^+ \wedge \neg(g^+) \supset ((f \wedge \text{more}) ; f^+) \wedge \neg((g \wedge \text{more}) ; g^+)$	1, 2, 3, Prop
5	$\vdash f \wedge \text{more} \supset g \wedge \text{more}$	1, Prop
6	$\vdash (f \wedge \text{more}) ; f^+ \supset (g \wedge \text{more}) ; f^+$	5, LeftChopImpChop
7	$\vdash f^+ \wedge \neg(g^+) \supset ((g \wedge \text{more}) ; f^+) \wedge \neg((g \wedge \text{more}) ; g^+)$	4, 6, Prop
8	$\vdash g \wedge \text{more} \supset \text{more}$	Prop
9	$\vdash f^+ \supset g^+$	7, 8, ChopContra

$$\vdash f \equiv g \Rightarrow \vdash f^+ \equiv g^+$$

ChopPlusEqvChopPlus

Proof:

1	$\vdash f \equiv g$	given
2	$\vdash f \supset g$	1, Prop
3	$\vdash f^+ \supset g^+$	2, ChopPlusImpChopPlus
4	$\vdash g \supset f$	1, Prop
5	$\vdash g^+ \supset f^+$	4, ChopPlusImpChopPlus
6	$\vdash f^+ \equiv g^+$	3, 5, Prop

8 Properties of *chop-star*

$$\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}) ; f^*$$

CSEqv

Proof:

- | | | |
|---|---|----------------------|
| 1 | $\vdash f^+ \equiv f \vee (f \wedge \text{more}); f^+$ | ChopPlusEqv |
| 2 | $\vdash \text{empty} \equiv \neg \text{more}$ | PTL |
| 3 | $\vdash \text{empty} \vee f^+ \equiv \text{empty} \vee (f \wedge \text{more}) \vee (f \wedge \text{more}); f^+$ | 1, 2, Prop |
| 4 | $\vdash (f \wedge \text{more}); \text{empty} \equiv f \wedge \text{more}$ | ChopEmpty |
| 5 | $\vdash (f \wedge \text{more}); (\text{empty} \vee f^+) \equiv (f \wedge \text{more}); \text{empty} \vee (f \wedge \text{more}); f^+$ | ChopOrEqv |
| 6 | $\vdash \text{empty} \vee f^+ \equiv \text{empty} \vee ((f \wedge \text{more}); (\text{empty} \vee f^+))$ | 3, 4, 5, Prop |
| 7 | $\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}); f^*$ | 6, def. of * |
-

$\vdash \text{empty} \supset f^*$

EmptyImpCS

Proof:

- | | | |
|---|---|----------------|
| 1 | $\vdash f^* \equiv \text{empty} \vee f^+$ | def. of * |
| 2 | $\vdash \text{empty} \supset f^*$ | 1, Prop |
- Here is an straightforward corollary:
-

$\vdash \neg f^* \supset \text{more}$

NotCSImpMore

$\vdash f^+ \supset f^*$

ChopPlusImpCS

Proof:

- | | | |
|---|--|--------------|
| 1 | $\vdash f^+ \supset \text{empty} \vee f^+$ | Prop |
| 2 | $\vdash f^+ \supset f^*$ | 1, def. of * |
-

$\vdash f \supset f^*$

ImpCS

Proof:

- | | | |
|---|--|--------------------|
| 1 | $\vdash f \supset f^+$ | ImpChopPlus |
| 2 | $\vdash f \supset \text{empty} \vee f^+$ | 1, Prop |
| 3 | $\vdash f \supset f^*$ | 2, def. of * |
-

$$\vdash f^*; g \equiv g \vee f^+; g$$

CSChopEqvOrChopPlusChop

Proof:

- 1 $\vdash f^* \equiv \text{empty} \vee f^+$ def. of $*$
- 2 $\vdash f^*; g \equiv g \vee f^+; g$ 1, **EmptyOrChopEqvRule**

$$\vdash f^* \equiv \text{empty} \vee (f; f^*)$$

CSEqvOrChopCS

Proof for \supset :

- 1 $\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}); f^*$ **CSEqv**
- 2 $\vdash (f \wedge \text{more}); f^* \supset f; f^*$ **AndChopA**
- 3 $\vdash f^* \supset \text{empty} \vee f; f^*$ 1, 2, **Prop**

Proof for \subset :

- 1 $\vdash \text{empty} \supset f^*$ **EmptyImpCS**
- 2 $\vdash \text{empty} \vee \text{more}$ **PTL**
- 3 $\vdash f \supset \text{empty} \vee (f \wedge \text{more})$ 2, **Prop**
- 4 $\vdash f; f^* \supset f^* \vee (f \wedge \text{more}); f^*$ 3, **EmptyOrChopImpRule**
- 5 $\vdash f^* \equiv f \vee (f \wedge \text{more}); f^*$ **CSEqv**
- 6 $\vdash (f \wedge \text{more}); f^* \supset f^*$ 5, **Prop**
- 7 $\vdash \text{empty} \vee (f; f^*) \supset f^*$ 1, 6, **Prop**

$$\vdash f; f^* \supset f^*$$

ChopCSImpCS

Proof:

- 1 $\vdash f^* \equiv \text{empty} \vee (f; f^*)$ **CSEqvOrChopCS**
- 2 $\vdash f; f^* \supset f^*$ 1, **Prop**

$$\vdash f^* \wedge \text{more} \supset f^+$$

CSAndMoreImpChopPlus

Proof:

- 1 $\vdash f^* \wedge \text{more} \supset f; f^*$ **CSAndMoreImpChopCS**
- 2 $\vdash f^* \wedge \text{more} \supset f^+$ 1, def. of $+$

$$\vdash f^* \wedge \text{more} \equiv (f \wedge \text{more}); f^*$$

CSAndMoreEqvAndMoreChop

Proof for \supset :

- 1 $\vdash (\text{empty} \vee (f \wedge \text{more}); f^*) \wedge \text{more} \supset (f \wedge \text{more}); f^*$ **PTL**
- 2 $\vdash f^* \wedge \text{more} \supset (f \wedge \text{more}); f^*$ 1, def. of $*$

Proof for \subset :

- 1 $\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}); f^*$ **CSEqv**
 - 2 $\vdash (f \wedge \text{more}); f^* \supset f^*$ **1, Prop**
 - 3 $\vdash (f \wedge \text{more}); f^* \supset \text{more}$ **MoreChopImpMore**
 - 4 $\vdash (f \wedge \text{more}); f^* \supset f^* \wedge \text{more}$ **2, 3, Prop**
-

$$\vdash f^* \wedge \text{more} \supset f; f^*$$

[CSAndMoreImpChopCS](#)

Proof:

- 1 $\vdash f^* \wedge \text{more} \equiv (f \wedge \text{more}); f^*$ **CSAndMoreEqvAndMoreChop**
 - 2 $\vdash (f \wedge \text{more}); f^* \supset f; f^*$ **AndChopA**
 - 3 $\vdash f^* \wedge \text{more} \supset f; f^*$ **1, 2, Prop**
-

$$\vdash f^* \wedge \text{more} \supset f^*; f$$

[CSAndMoreImpCSChop](#)

Proof:

- 1 $\vdash f^* \wedge \text{more} \equiv (f \wedge \text{more}); f^*$ **CSAndMoreEqvAndMoreChop**
- 2 $\vdash \text{empty} \vee \text{more}$ **PTL**
- 3 $\vdash f^* \supset \text{empty} \vee (f^* \wedge \text{more})$ **2, Prop**
- 4 $\vdash (f \wedge \text{more}); f^* \supset (f \wedge \text{more}) \vee ((f \wedge \text{more}); (f^* \wedge \text{more}))$ **3, ChopEmptyOrImpRule**
- 5 $\vdash f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}); (f^* \wedge \text{more})$ **CSMoreNotImpChopCSAndMore**
- 6 $\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}); f^*$ **CSEqv**
- 7 $\vdash f^*; f \equiv f \vee ((f \wedge \text{more}); f^*); f$ **6, EmptyOrChopEqvRule**
- 8 $\vdash ((f \wedge \text{more}); f^*); f \equiv (f \wedge \text{more}); (f^*; f)$ **ChopAssoc**
- 9 $\vdash (f^* \wedge \text{more}) \wedge \neg(f^*; f) \supset$
 $(f \wedge \text{more}); (f^* \wedge \text{more}) \wedge \neg((f \wedge \text{more}); (f^*; f))$ **5, 7, 8, Prop**
- 10 $\vdash f \wedge \text{more} \supset \text{more}$ **Prop**
- 11 $\vdash f^* \wedge \text{more} \supset f^*; f$ **9, 10, ChopContra**

The following lemma is used in the proof of **CSAndMoreImpCSChop**:

$$\vdash f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}); (f^* \wedge \text{more})$$

[CSMoreNotImpChopCSAndMore](#)

Proof:

1	$\vdash f^* \wedge \text{more} \equiv (f \wedge \text{more}); f^*$	CSAndMoreEqvAndMoreChop
2	$\vdash \text{empty} \vee \text{more}$	PTL
3	$\vdash f^* \supset \text{empty} \vee (f^* \wedge \text{more})$	2, Prop
4	$\vdash (f \wedge \text{more}); f^* \supset (f \wedge \text{more}) \vee ((f \wedge \text{more}); (f^* \wedge \text{more}))$	3, ChopEmptyOrImpRule
5	$\vdash (f \wedge \text{more}); f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}); (f^* \wedge \text{more})$	4, Prop
6	$\vdash f^* \wedge \text{more} \wedge \neg f \supset (f \wedge \text{more}); (f^* \wedge \text{more})$	1, 5, Prop

$\vdash f^*; f^* \supset f^*$ CSChopCSImpCS

Proof:

1	$\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}); f^*$	CSEqv
2	$\vdash f^*; f^* \equiv f^* \vee ((f \wedge \text{more}); f^*); f^*$	1, EmptyOrChopEqvRule
3	$\vdash f^*; f^* \wedge \neg(f^*) \supset ((f \wedge \text{more}); f^*); f^* \wedge \neg((f \wedge \text{more}); f^*)$	1, 2, Prop
4	$\vdash ((f \wedge \text{more}); f^*); f^* \equiv (f \wedge \text{more}); f^*; f^*$	ChopAssoc
5	$\vdash f^*; f^* \wedge \neg(f^*) \supset (f \wedge \text{more}); f^*; f^* \wedge \neg((f \wedge \text{more}); f^*)$	3, 4, Prop
6	$\vdash f \wedge \text{more} \supset \text{more}$	Prop
7	$\vdash f^*; f^* \supset f^*$	5, 6, ChopContra

$\vdash f^*; f \supset f^*$ CSChopImpCS

Proof:

1	$\vdash f \supset f^*$	ImpCS
2	$\vdash f^*; f \supset f^*; f^*$	1, LeftChopImpChop
3	$\vdash f^*; f^* \supset f^*$	CSChopCSImpCS
4	$\vdash f^*; f \supset f^*$	2, 3, ImpChain

$\vdash (f^*)^* \supset f^*$ CSCSImpCS

Proof:

1	$\vdash \text{empty} \supset f^*$	EmptyImpCS
2	$\vdash (f^* \wedge \text{more}); f^* \supset f^*; f^*$	AndChopA
3	$\vdash f^*; f^* \supset f^*$	CSChopCSImpCS
4	$\vdash (f^* \wedge \text{more}); f^* \supset f^*$	2, 3, ImpChain
5	$\vdash (f^*)^* \supset f^*$	1, 4, CSElim

$\vdash f \supset g \Rightarrow \vdash f^* \supset g^*$ CSImpCS

Proof:

- | | | |
|---|--|-------------------------------|
| 1 | $\vdash f \supset g$ | given |
| 2 | $\vdash f^+ \supset g^+$ | 1, ChopPlusImpChopPlus |
| 3 | $\vdash \text{empty} \vee f^+ \supset \text{empty} \vee g^+$ | 2, Prop |
| 4 | $\vdash f^* \supset g^*$ | 3, def. of * |
-

$$\vdash f \equiv g \Rightarrow \vdash f^* \equiv g^*$$

CSEqvCS

Proof:

- | | | |
|---|---|-------------------------------|
| 1 | $\vdash f \equiv g$ | given |
| 2 | $\vdash f^+ \equiv g^+$ | 1, ChopPlusEqvChopPlus |
| 3 | $\vdash \text{empty} \vee f^+ \equiv \text{empty} \vee g^+$ | 2, Prop |
| 4 | $\vdash f^* \equiv g^*$ | 3, def. of * |
-

$$\vdash (f \wedge g)^* \supset f^*$$

AndCSA

Proof:

- | | | |
|---|-------------------------------------|-------------------|
| 1 | $\vdash f \wedge g \supset f$ | Prop |
| 2 | $\vdash (f \wedge g)^* \supset f^*$ | 1, CSImpCS |
-

$$\vdash (f \wedge g)^* \supset g^*$$

AndCSB

Proof:

- | | | |
|---|-------------------------------------|-------------------|
| 1 | $\vdash f \wedge g \supset g$ | Prop |
| 2 | $\vdash (f \wedge g)^* \supset g^*$ | 1, CSImpCS |
-

$$\vdash f \wedge \text{more} \supset (g \wedge \text{more}); f \Rightarrow \vdash f \supset g^*$$

CSIntro

Proof:

- | | | |
|---|---|-------------------------|
| 1 | $\vdash f \wedge \text{more} \supset (g \wedge \text{more}); f$ | given |
| 2 | $\vdash \text{more} \equiv \neg \text{empty}$ | PTL |
| 3 | $\vdash f \wedge \neg \text{empty} \supset (g \wedge \text{more}); f$ | 1, 2, Prop |
| 4 | $\vdash g^* \equiv \text{empty} \vee (g \wedge \text{more}); g^*$ | CSEqv |
| 5 | $\vdash f \wedge \neg g^* \supset (g \wedge \text{more}); f \wedge \neg((g \wedge \text{more}); g^*)$ | 3, 4, Prop |
| 6 | $\vdash g \wedge \text{more} \supset \text{more}$ | Prop |
| 7 | $\vdash f \supset g^*$ | 5, 6, ChopContra |

$$\vdash \text{empty} \supset g, \vdash (f \wedge \text{more}); g \supset g \Rightarrow \vdash f^* \supset g$$

CSElim

Proof:

1	$\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}); f^*$	CSEqv
2	$\vdash \text{empty} \supset g$	given
3	$\vdash (f \wedge \text{more}); g \supset g$	given
4	$\vdash f^* \wedge \neg g \supset (f \wedge \text{more}); f^* \wedge \neg((f \wedge \text{more}); g)$	1, 2, 3, Prop
5	$\vdash f \wedge \text{more} \supset \text{more}$	Prop
6	$\vdash f^* \supset g$	4, 5, ChopContra

$$\vdash \text{empty} \supset g, \vdash f; g \supset g \Rightarrow \vdash f^* \supset g$$

CSElimWithoutMore

Proof:

1	$\vdash \text{empty} \supset g$	given
2	$\vdash f; g \supset g$	given
3	$\vdash (f \wedge \text{more}); g \supset f; g$	AndChopA
4	$\vdash (f \wedge \text{more}); g \supset g$	2, 3, ImpChain
5	$\vdash f^* \supset g$	1, 4, CSElim

$$\vdash f \equiv g^*; h \Rightarrow \vdash f \equiv (g; f) \vee h$$

CSChopEqvChopOrRule

Proof:

1	$\vdash f \equiv g^*; h$	given
2	$\vdash g^* \equiv \text{empty} \vee (g; g^*)$	CSEqvOrChopCS
3	$\vdash g^*; h \equiv h \vee ((g; g^*); h)$	2, EmptyOrChopEqvRule
4	$\vdash (g; g^*); h \equiv g; (g^*; h)$	ChopAssoc
5	$\vdash g; f \equiv g; (g^*; h)$	1, RightChopEqvChop
6	$\vdash f \equiv (g; f) \vee h$	1, 3, 4, 5, Prop

$$\vdash f \wedge \neg h \supset g; f, \vdash g \supset \text{more} \Rightarrow \vdash f \supset g^*; h$$

CSChopIntroRule

Proof:

1	$\vdash f \wedge \neg h \supset g ; f$	given
2	$\vdash g \supset \text{more}$	given
3	$\vdash g \supset g \wedge \text{more}$	2, Prop
4	$\vdash g ; f \supset (g \wedge \text{more}) ; f$	3, LeftChopImpChop
5	$\vdash f \supset (g \wedge \text{more}) ; f \vee h$	1, 4, Prop
6	$\vdash g^* \equiv \text{empty} \vee (g \wedge \text{more}) ; g^*$	CSEqv
7	$\vdash g^* ; h \equiv h \vee ((g \wedge \text{more}) ; g^*) ; h$	6, EmptyOrChopImpRule
8	$\vdash ((g \wedge \text{more}) ; g^*) ; h \equiv (g \wedge \text{more}) ; (g^* ; h)$	ChopAssoc
9	$\vdash g^* ; h \equiv h \vee (g \wedge \text{more}) ; (g^* ; h)$	7, 8, Prop
10	$\vdash f \wedge \neg(g^* ; h) \supset (g \wedge \text{more}) ; f \wedge \neg((g \wedge \text{more}) ; (g^* ; h))$	5, 9, Prop
11	$\vdash g \wedge \text{more} \supset \text{more}$	Prop
12	$\vdash f \supset g^* ; h$	10, 11, ChopContra

$\vdash f \supset \text{empty} \vee (\Box w \wedge \text{more}) ; f \Rightarrow \vdash w \wedge f \supset \Box w$

CSImpBox

Proof:

1	$\vdash f \supset \text{empty} \vee (\Box w \wedge \text{more}) ; f$	given
2	$\vdash w \wedge \neg \Box w \supset \neg \text{empty}$	PTL
3	$\vdash w \wedge f \wedge \neg \Box w \supset (\Box w \wedge \text{more}) ; f$	1, 2, Prop
4	$\vdash \Box w \wedge \text{more} \supset (\Box w \wedge \text{more}) \wedge \text{fin } w$	PTL
5	$\vdash (\Box w \wedge \text{more}) ; f \supset ((\Box w \wedge \text{more}) \wedge \text{fin } w) ; f$	4, LeftChopImpChop
6	$\vdash ((\Box w \wedge \text{more}) \wedge \text{fin } w) ; f \equiv (\Box w \wedge \text{more}) ; (w \wedge f)$	AndFinChopEqvStateAndChop
7	$\vdash \neg \Box w \supset (\Box w) \rightsquigarrow \neg \Box w$	NotBoxStateImpBoxYieldsNo
8	$\vdash (\Box w) \rightsquigarrow \neg \Box w \supset (\Box w \wedge \text{more}) \rightsquigarrow \neg \Box w$	AndYieldsA
9	$\vdash (\Box w \wedge \text{more}) ; (w \wedge f) \wedge (\Box w \wedge \text{more}) \rightsquigarrow \neg \Box w \supset$ $(\Box w \wedge \text{more}) ; ((w \wedge f) \wedge \neg \Box w)$	ChopAndYieldsImp
10	$\vdash (w \wedge f) \wedge \neg \Box w \supset (\Box w \wedge \text{more}) ; ((w \wedge f) \wedge \neg \Box w)$	3, 5, 6, 7, 8, 9, Prop
11	$\vdash (\Box w \wedge \text{more}) ; ((w \wedge f) \wedge \neg \Box w) \supset \text{more} ; ((w \wedge f) \wedge \neg \Box w)$	AndChopB
12	$\vdash (w \wedge f) \wedge \neg \Box w \supset \text{more} ; ((w \wedge f) \wedge \neg \Box w)$	10, 11, ImpChain
13	$\vdash w \wedge f \supset \Box w$	MoreChopContra

$\vdash w \wedge (\Box w)^* \equiv \Box w$

BoxCSEqvBox

Proof for \supset :

1	$\vdash (\Box w)^* \equiv \text{empty} \vee (\Box w \wedge \text{more}) ; (\Box w)^*$	CSEqv
2	$\vdash (\Box w)^* \supset \text{empty} \vee (\Box w \wedge \text{more}) ; (\Box w)^*$	1, Prop
3	$\vdash w \wedge (\Box w)^* \supset \Box w$	2, CSImpBox

Proof for \subset :

1	$\vdash \Box w \supset w$	PTL
2	$\vdash \Box w \supset (\Box w)^*$	ImpCS
3	$\vdash \Box w \supset w \wedge (\Box w)^*$	1, 2, Prop

$$\vdash \Box w \wedge f^* \equiv w \wedge (\Box w \wedge f)^*$$

BoxStateAndCSEqvCS

Proof for \supset :

1	$\vdash \Box w \supset w$	PTL
2	$\vdash f^* \wedge \text{more} \equiv (f \wedge \text{more}) ; f^*$	CSAndMoreEqvAndMoreChop
3	$\vdash \Box w \wedge ((f \wedge \text{more}) ; f^*) \equiv (\Box w \wedge f \wedge \text{more}) ; (\Box w \wedge f^*)$	BoxStateAndChopEqvChop
4	$\vdash \Box w \wedge f \wedge \text{more} \supset (\Box w \wedge f) \wedge \text{more}$	PTL
5	$\vdash (\Box w \wedge f \wedge \text{more}) ; (\Box w \wedge f^*) \supset$ $((\Box w \wedge f) \wedge \text{more}) ; (\Box w \wedge f^*)$	4, LeftChopImpChop
6	$\vdash (\Box w \wedge f^*) \wedge \text{more} \supset ((\Box w \wedge f) \wedge \text{more}) ; (\Box w \wedge f^*)$	2, 3, 5, Prop
7	$\vdash \Box w \wedge f^* \supset (\Box w \wedge f)^*$	6, CSIntro
8	$\vdash \Box w \wedge f^* \supset w \wedge (\Box w \wedge f)^*$	1, 7, Prop

Proof for \subset :

1	$\vdash (\Box w \wedge f)^* \supset (\Box w)^*$	AndCSA
2	$\vdash w \wedge (\Box w)^* \equiv \Box w$	BoxCSEqvBox
3	$\vdash (\Box w \wedge f)^* \supset f^*$	AndCSB
4	$\vdash w \wedge (\Box w \wedge f)^* \supset \Box w \wedge f^*$	1, 2, 3, Prop

See also the lemma **BoxStateAndChopEqvChop** for *chop*.

$$\vdash \boxplus(f \supset g) \supset f^* \supset g^*$$

BaCSImpCS

Proof:

1	$\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}) ; f^*$	CSEqv
2	$\vdash g^* \equiv \text{empty} \vee (g \wedge \text{more}) ; g^*$	CSEqv
3	$\vdash f^* \wedge \neg(g^*) \supset (f \wedge \text{more}) ; f^* \wedge \neg((g \wedge \text{more}) ; g^*)$	1, 2, Prop
4	$\vdash (f \supset g) \supset (f \wedge \text{more} \supset g \wedge \text{more})$	Prop
5	$\vdash \boxplus(f \supset g) \supset \boxplus(f \wedge \text{more} \supset g \wedge \text{more})$	4, BalmpBa
6	$\vdash \boxplus(f \wedge \text{more} \supset g \wedge \text{more}) \supset (f \wedge \text{more}) ; f^* \supset$ $(g \wedge \text{more}) ; f^*$	BaLeftChopImpChop
7	$\vdash \boxplus(f \supset g) \wedge (f \wedge \text{more}) ; f^* \supset (g \wedge \text{more}) ; f^*$	5, 6, Prop
8	$\vdash (g \wedge \text{more}) ; f^* \wedge \neg((g \wedge \text{more}) ; g^*) \supset$ $(g \wedge \text{more}) ; (f^* \wedge \neg(g^*))$	ChopAndNotChopImp
9	$\vdash (g \wedge \text{more}) ; (f^* \wedge \neg(g^*)) \supset \text{more} ; (f^* \wedge \neg(g^*))$	AndChopB
10	$\vdash \boxplus(f \supset g) \supset \text{more} ; (f^* \wedge \neg(g^*)) \supset$ $\text{more} ; (\boxplus(f \supset g) \wedge f^* \wedge \neg(g^*))$	BaChopImpChopBa
11	$\vdash \boxplus(f \supset g) \wedge f^* \wedge \neg(g^*) \supset$ $\text{more} ; (\boxplus(f \supset g) \wedge f^* \wedge \neg(g^*))$	3, 7, 8, 9, 10, Prop
12	$\vdash \neg(\boxplus(f \supset g) \wedge f^* \wedge \neg(g^*))$	11, MoreChopLoop
13	$\vdash \boxplus(f \supset g) \supset f^* \supset g^*$	12, Prop

The following corollary can be readily verified:

$$\vdash \Box(f \equiv g) \supset f^* \equiv g^*$$

BaCSEqvCS

$$\vdash \Box f \wedge g^* \supset (f \wedge g)^*$$

BaAndCSImport

Proof:

- | | | |
|---|--|-------------------|
| 1 | $\vdash f \supset (g \supset f \wedge g)$ | Prop |
| 2 | $\vdash \Box f \supset \Box(g \supset f \wedge g)$ | 1, BalmpBa |
| 3 | $\vdash \Box(g \supset f \wedge g) \supset g^* \supset (f \wedge g)^*$ | BaCSImpCS |
| 4 | $\vdash \Box f \wedge g^* \supset (f \wedge g)^*$ | 2, 3, Prop |

9 Properties of While

$$\vdash \text{while } w \text{ do } f \equiv \text{if } w \text{ then } f ; (\text{while } w \text{ do } f) \text{ else empty}$$

WhileEqvIf

Proof:

- | | | |
|---|---|-------------------------------|
| 1 | $\vdash \text{while } w \text{ do } f \equiv (w \wedge f)^* \wedge \text{fin} \neg w$ | def. of while |
| 2 | $\vdash (w \wedge f)^* \equiv \text{empty} \vee ((w \wedge f) ; (w \wedge f)^*)$ | CSEqvOrChopCS |
| 3 | $\vdash \text{empty} \wedge \text{fin} \neg w \equiv \neg w \wedge \text{empty}$ | PTL |
| 4 | $\vdash (w \wedge f) ; (w \wedge f)^* \equiv w \wedge f ; (w \wedge f)^*$ | StateAndChop |
| 5 | $\vdash (f ; (w \wedge f)^*) \wedge \text{fin} \neg w \equiv f ; ((w \wedge f)^* \wedge \text{fin} \neg w)$ | ChopAndFin |
| 6 | $\vdash f ; ((w \wedge f)^* \wedge \text{fin} \neg w) \equiv f ; \text{while } w \text{ do } f$ | def. of while |
| 7 | $\vdash \text{while } w \text{ do } f$
$\equiv (\neg w \wedge \text{empty}) \vee (w \wedge (f ; \text{while } w \text{ do } f))$ | 1, 2, 3, 4, 5, 6, Prop |
| 8 | $\vdash \text{while } w \text{ do } f \equiv \text{if } w \text{ then } f ; (\text{while } w \text{ do } f) \text{ else empty}$ | 7, Prop |

$$\vdash (\text{while } w \text{ do } f) ; g \equiv \text{if } w \text{ then } f ; (\text{while } w \text{ do } f) ; g \text{ else } g$$

WhileChopEqvIf

Proof:

- | | | |
|---|--|-------------------------|
| 1 | $\vdash \text{while } w \text{ do } f \equiv \text{if } w \text{ then } f ; (\text{while } w \text{ do } f) \text{ else empty}$ | WhileEqvIf |
| 2 | $\vdash (\text{while } w \text{ do } f) ; g \equiv$
$\text{if } w \text{ then } (f ; \text{while } w \text{ do } f) ; g \text{ else empty} ; g$ | 1, IfChopEqvRule |
| 3 | $\vdash (f ; \text{while } w \text{ do } f) ; g \equiv f ; (\text{while } w \text{ do } f) ; g$ | ChopAssoc |
| 4 | $\vdash \text{empty} ; g \equiv g$ | EmptyChop |
| 5 | $\vdash (\text{while } w \text{ do } f) ; g \equiv \text{if } w \text{ then } f ; (\text{while } w \text{ do } f) ; g \text{ else } g$ | 2, 3, 4, Prop |

$$\vdash f \equiv (\text{while } w \text{ do } g); h \Rightarrow f \equiv \text{if } w \text{ then } g; f \text{ else } h$$

WhileChopEqvIfRule

Proof:

- | | | |
|---|---|----------------------------|
| 1 | $\vdash f \equiv (\text{while } w \text{ do } g); h$ | given |
| 2 | $\vdash (\text{while } w \text{ do } g); h \equiv \text{if } w \text{ then } g; (\text{while } w \text{ do } g); h \text{ else } h$ | WhileChopEqvIf |
| 3 | $\vdash g; f \equiv g; (\text{while } w \text{ do } g); h$ | 1, RightChopEqvChop |
| 4 | $\vdash f \equiv \text{if } w \text{ then } g; f \text{ else } h$ | 1, 2, 3, Prop |
-

$$\vdash \text{while } w \text{ do } f \supset \text{fin } \neg w$$

WhileImpFin

Proof:

- | | | |
|---|--|------------------|
| 1 | $\vdash (w \wedge f)^* \wedge \text{fin } \neg w \supset \text{fin } \neg w$ | Prop |
| 2 | $\vdash \text{while } w \text{ do } f \supset \text{fin } \neg w$ | 1, def. of while |
-

$$\vdash \text{while } w \text{ do } f \equiv (\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}); \text{while } w \text{ do } f)$$

WhileEqvEmptyOrChopWhile

Proof:

- | | | |
|----|---|---------------------------|
| 1 | $\vdash (w \wedge f)^* \equiv \text{empty} \vee ((w \wedge f) \wedge \text{more}); (w \wedge f)^*$ | CSEqv |
| 2 | $\vdash (w \wedge f) \wedge \text{more} \equiv w \wedge (f \wedge \text{more})$ | Prop |
| 3 | $\vdash ((w \wedge f) \wedge \text{more}); (w \wedge f)^* \equiv (w \wedge f \wedge \text{more}); (w \wedge f)^*$ | 2, LeftChopEqvChop |
| 4 | $\vdash (w \wedge f)^* \equiv \text{empty} \vee (w \wedge f \wedge \text{more}); (w \wedge f)^*$ | 1, 3, Prop |
| 5 | $\vdash (w \wedge f)^* \wedge \text{fin } \neg w \equiv$
$(\text{empty} \wedge \text{fin } \neg w) \vee ((w \wedge f \wedge \text{more}); (w \wedge f)^* \wedge \text{fin } \neg w)$ | 1, Prop |
| 6 | $\vdash \text{empty} \wedge \text{fin } \neg w \equiv \neg w \wedge \text{empty}$ | PTL |
| 7 | $\vdash (w \wedge f \wedge \text{more}); (w \wedge f)^* \equiv w \wedge (f \wedge \text{more}); (w \wedge f)^*$ | StateAndChop |
| 8 | $\vdash (f \wedge \text{more}); (w \wedge f)^* \wedge \text{fin } \neg w \equiv$
$(f \wedge \text{more}); ((w \wedge f)^* \wedge \text{fin } \neg w)$ | ChopAndFin |
| 9 | $\vdash (w \wedge f)^* \wedge \text{fin } \neg w \equiv$
$(\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}); ((w \wedge f)^* \wedge \text{fin } \neg w))$ | 5, 6, 7, 8, Prop |
| 10 | $\vdash \text{while } w \text{ do } f \equiv$
$(\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}); \text{while } w \text{ do } f)$ | 9, def. of while |
-

$$\vdash \neg w \wedge f \supset \text{empty}$$

$$\vdash w \wedge f \supset (g \wedge \text{more}); f$$

$$\Rightarrow \vdash f \supset \text{while } w \text{ do } g$$

WhileIntro

Proof:

- 1 $\vdash \neg w \wedge f \supset \text{empty}$ given
 - 2 $\vdash w \wedge f \supset (g \wedge \text{more}) ; f$ given
 - 3 $\vdash \text{while } w \text{ do } g \equiv$
 $(\neg w \wedge \text{empty}) \vee (w \wedge (g \wedge \text{more}) ; \text{while } w \text{ do } g)$ **WhileEqvEmptyOrChopWhile**
 - 4 $\vdash f \wedge \neg \text{while } w \text{ do } g \supset$
 $(g \wedge \text{more}) ; f \wedge \neg((g \wedge \text{more}) ; \text{while } w \text{ do } g)$ 1, 2, 3, **Prop**
 - 5 $\vdash g \wedge \text{more} \supset \text{more}$ **Prop**
 - 6 $\vdash f \supset \text{while } w \text{ do } g$ 4, 5, **ChopContra**
-

- $\vdash \neg w \wedge \text{empty} \supset g$ WhileElim
- $\vdash w \wedge (f \wedge \text{more}) ; g \supset g$
- $\Rightarrow \vdash \text{while } w \text{ do } f \supset g$
-

Proof:

- 1 $\vdash \text{while } w \text{ do } f \equiv$
 $(\neg w \wedge \text{empty}) \vee (w \wedge (f \wedge \text{more}) ; \text{while } w \text{ do } f)$ **WhileEqvEmptyOrChopWhile**
 - 2 $\vdash \neg w \wedge \text{empty} \supset g$ given
 - 3 $\vdash w \wedge (f \wedge \text{more}) ; g \supset g$ given
 - 4 $\vdash \text{while } w \text{ do } f \wedge \neg g \supset$
 $(f \wedge \text{more}) ; \text{while } w \text{ do } f \wedge \neg((f \wedge \text{more}) ; g)$ 1, 2, 3, **Prop**
 - 5 $\vdash f \wedge \text{more} \supset \text{more}$ **Prop**
 - 6 $\vdash \text{while } w \text{ do } f \supset g$ 4, 5, **ChopContra**
-

- $\vdash \boxplus(f \supset g) \supset (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$ BaWhileImpWhile
-

Proof:

- 1 $\vdash (f \supset g) \supset ((w \wedge f) \supset (w \wedge g))$ **Prop**
 - 2 $\vdash \boxplus(f \supset g) \supset \boxplus((w \wedge f) \supset (w \wedge g))$ 1, **BalmpBa**
 - 3 $\vdash \boxplus((w \wedge f) \supset (w \wedge g)) \supset ((w \wedge f)^* \supset (w \wedge g)^*)$ **BaCSImpCS**
 - 4 $\vdash \boxplus(f \supset g) \supset ((w \wedge f)^* \wedge \text{fin } \neg w \supset (w \wedge g)^* \wedge \text{fin } \neg w)$ 2, 3, **Prop**
 - 5 $\vdash \boxplus(f \supset g) \supset (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$ 4, def. of while
-

- $\vdash f \supset g \Rightarrow \vdash (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$ WhileImpWhile
-

Proof:

1	$\vdash f \supset g$	given
2	$\vdash \Box(f \supset g)$	1, BaGen
3	$\vdash \Box(f \supset g) \supset (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$	BaWhileImpWhile
4	$\vdash (\text{while } w \text{ do } f) \supset (\text{while } w \text{ do } g)$	2, 3, MP

10 Properties of Halt

$\vdash \text{halt } w ; f \equiv \text{if } w \text{ then } f \text{ else } \circ(\text{halt } w ; f)$ HaltChopEqv

Proof:

1	$\vdash \text{halt } w \equiv \text{if } w \text{ then empty else } \circ \text{halt } w$	PTL
2	$\vdash \text{halt } w ; f \equiv \text{if } w \text{ then empty ; } f \text{ else } (\circ \text{halt } w) ; f$	1, IfChopEqvRule
3	$\vdash \text{empty ; } f \equiv f$	EmptyChop
4	$\vdash (\circ \text{halt } w) ; f \equiv \circ(\text{halt } w ; f)$	NextChop
5	$\vdash \text{halt } w ; f \equiv \text{if } w \text{ then } f \text{ else } \circ(\text{halt } w ; f)$	2, 3, 4, Prop

$\vdash w \wedge (\text{halt } w ; f) \supset f$ AndHaltChopImp

Proof:

1	$\vdash \text{halt } w ; f \equiv \text{if } w \text{ then } f \text{ else } \circ(\text{halt } w ; f)$	HaltChopEqv
2	$\vdash w \wedge (\text{halt } w ; f) \supset f$	1, Prop

$\vdash \neg w \wedge (\text{halt } w ; f) \supset \circ(\text{halt } w ; f)$ NotAndHaltChopImpNext

Proof:

1	$\vdash \text{halt } w ; f \equiv \text{if } w \text{ then } f \text{ else } \circ(\text{halt } w ; f)$	HaltChopEqv
2	$\vdash \neg w \wedge (\text{halt } w ; f) \supset \circ(\text{halt } w ; f)$	1, Prop

$\vdash \neg w \wedge (\text{halt } w ; f) \supset \text{skip } \rightsquigarrow (\text{halt } w ; f)$ NotAndHaltChopImpSkipYields

Proof:

1	$\vdash \neg w \wedge (\text{halt } w ; f) \supset \circ(\text{halt } w ; f)$	NotAndHaltChopImpNext
2	$\vdash \circ(\text{halt } w ; f) \supset \text{skip } \rightsquigarrow (\text{halt } w ; f)$	NextImpSkipYields
3	$\vdash \neg w \wedge (\text{halt } w ; f) \supset \text{skip } \rightsquigarrow (\text{halt } w ; f)$	1, 2, ImpChain

$\vdash \text{halt } w ; f \supset \neg(\text{halt } w ; \neg f)$

[HaltChopImpNotHaltChopNot](#)

Proof:

- | | | |
|---|---|----------------------|
| 1 | $\vdash \text{halt } w ; f \equiv \text{if } w \text{ then } f \text{ else } \text{O}(\text{halt } w ; f)$ | HaltChopEqv |
| 2 | $\vdash \text{halt } w ; \neg f \equiv \text{if } w \text{ then } \neg f \text{ else } \text{O}(\text{halt } w ; \neg f)$ | HaltChopEqv |
| 3 | $\vdash (\text{halt } w ; f) \wedge (\text{halt } w ; \neg f) \supset \text{O}(\text{halt } w ; f) \wedge \neg \text{O}(\text{halt } w ; \neg f)$ | 1, 2, Prop |
| 4 | $\vdash \text{halt } w ; f \supset \neg(\text{halt } w ; \neg f)$ | 3, NextContra |

$\vdash \text{halt } w ; f \supset (\text{halt } w) \rightsquigarrow f$

[HaltChopImpHaltYields](#)

Proof:

- | | | |
|---|---|----------------------------------|
| 1 | $\vdash \text{halt } w ; f \supset \neg(\text{halt } w ; \neg f)$ | HaltChopImpNotHaltChopNot |
| 2 | $\vdash \text{halt } w ; f \supset (\text{halt } w) \rightsquigarrow f$ | 1, def. of \rightsquigarrow |

$\vdash (\text{halt } w) ; f \wedge (\text{halt } w) ; g \supset (\text{halt } w) ; (f \wedge g)$

[HaltChopAnd](#)

Proof:

- | | | |
|---|---|------------------------------|
| 1 | $\vdash (\text{halt } w) ; g \supset (\text{halt } w) \rightsquigarrow g$ | HaltChopImpHaltYields |
| 2 | $\vdash (\text{halt } w) ; f \wedge (\text{halt } w) ; g \supset (\text{halt } w) ; f \wedge (\text{halt } w) \rightsquigarrow g$ | 1, 2, Prop |
| 3 | $\vdash (\text{halt } w) ; f \wedge (\text{halt } w) \rightsquigarrow g \supset (\text{halt } w) ; (f \wedge g)$ | ChopAndYieldsImp |
| 4 | $\vdash (\text{halt } w) ; f \wedge (\text{halt } w) ; g \supset (\text{halt } w) ; (f \wedge g)$ | 2, 3, Prop |

$\vdash (\text{halt } w \wedge f) ; f_1 \wedge (\text{halt } w ; g) \supset (\text{halt } w \wedge f) ; (f_1 \wedge g)$

[HaltAndChopAndHaltChopImpHaltAndChopAnd](#)

Proof:

- | | | |
|---|--|----------------------------------|
| 1 | $\vdash f_1 \supset \neg g \vee (f_1 \wedge g)$ | Prop |
| 2 | $\vdash (\text{halt } w \wedge f) ; f_1 \supset$
$(\text{halt } w \wedge f) ; \neg g \vee ((\text{halt } w \wedge f) ; (f_1 \wedge g))$ | 1, ChopOrImpRule |
| 3 | $\vdash (\text{halt } w \wedge f) ; \neg g \supset \text{halt } w ; \neg g$ | AndChopA |
| 4 | $\vdash \text{halt } w ; g \supset \neg(\text{halt } w ; \neg g)$ | HaltChopImpNotHaltChopNot |
| 5 | $\vdash (\text{halt } w \wedge f) ; f_1 \wedge (\text{halt } w ; g) \supset (\text{halt } w \wedge f) ; (f_1 \wedge g)$ | 2, 3, 4, Prop |

$\vdash (\text{halt } w) ; f \supset (\Box \neg w) \rightsquigarrow ((\text{halt } w) ; f)$

[HaltImpBoxYields](#)

Proof:

1	$\vdash (\Box \neg w); \neg(\text{halt } w; f) \supset \Diamond(\Box \neg w)$	ChopImpDi
2	$\vdash \Box \neg w \supset \neg w$	PTL
3	$\vdash \Diamond(\Box \neg w) \supset \Diamond \neg w$	2, DilmpDi
4	$\vdash \Diamond \neg w \equiv \neg w$	DiState
5	$\vdash (\Box \neg w); \neg(\text{halt } w; f) \supset \neg w$	1, 2, 4, Prop
6	$\vdash \text{halt } w; f \equiv \text{if } w \text{ then } f \text{ else } \text{O}(\text{halt } w; f)$	HaltChopEqv
7	$\vdash (\text{halt } w; f) \wedge (\Box \neg w); \neg(\text{halt } w; f) \supset \text{O}((\text{halt } w); f)$	5, 6, Prop
8	$\vdash \Box \neg w \supset \text{empty} \vee \text{O} \Box \neg w$	PTL
9	$\vdash (\Box \neg w); \neg(\text{halt } w; f) \supset$ $\neg(\text{halt } w; f) \vee \text{O}((\Box \neg w); \neg(\text{halt } w; f))$	8, EmptyOrNextChopImpRule
10	$\vdash (\text{halt } w); f \wedge (\Box \neg w); \neg(\text{halt } w; f) \supset$ $\text{O}((\Box \neg w); \neg(\text{halt } w; f))$	9, Prop
11	$\vdash (\text{halt } w); f \wedge (\Box \neg w); \neg(\text{halt } w; f) \supset$ $\text{O}((\text{halt } w); f) \wedge \text{O}((\Box \neg w); \neg(\text{halt } w; f))$	7, 10, Prop
12	$\vdash \text{O}((\text{halt } w); f) \wedge \text{O}((\Box \neg w); \neg(\text{halt } w; f)) \supset$ $\text{O}(((\text{halt } w); f) \wedge ((\Box \neg w); \neg(\text{halt } w; f)))$	PTL
13	$\vdash (\text{halt } w); f \wedge (\Box \neg w); \neg(\text{halt } w; f) \supset$ $\text{O}(((\text{halt } w); f) \wedge ((\Box \neg w); \neg(\text{halt } w; f)))$	11, 12, ImpChain
14	$\vdash \neg((\text{halt } w); f \wedge (\Box \neg w); \neg(\text{halt } w; f))$	13, NextLoop
15	$\vdash (\text{halt } w); f \supset \neg((\Box \neg w); \neg(\text{halt } w; f))$	14, Prop
16	$\vdash (\text{halt } w); f \supset (\Box \neg w) \rightsquigarrow ((\text{halt } w); f)$	15, def. of \rightsquigarrow

11 Properties of groups of chops

$$\vdash (f_1; \dots; f_m); (g_1; \dots; g_n) \equiv f_1; \dots; f_m; g_1; \dots; g_n$$

ChopGroupMerge

We prove this by induction on m .

Proof for $m=1$:

$$1 \vdash f_1; (g_1; \dots; g_n) \equiv f_1; g_1; \dots; g_n \quad \text{def. of } \langle \text{chop} \rangle$$

Proof for $m > 1$:

$$1 \vdash (f_1; f_2; \dots; f_m); (g_1; \dots; g_n) \equiv$$

$$f_1; ((f_2; \dots; f_m); (g_1; \dots; g_n)) \quad \text{ChopAssoc}$$

$$2 \vdash (f_2; \dots; f_m); (g_1; \dots; g_n) \equiv$$

$$f_2; \dots; f_m; g_1; \dots; g_n \quad \text{induction hypothesis}$$

$$3 \vdash f_1; ((f_2; \dots; f_m); (g_1; \dots; g_n)) \equiv$$

$$f_1; (f_2; \dots; f_m; g_1; \dots; g_n) \quad 2, \text{RightChopEqvChop}$$

$$4 \vdash (f_1; f_2; \dots; f_m); (g_1; \dots; g_n) \equiv$$

$$f_1; f_2; \dots; f_m; g_1; \dots; g_n \quad 1, 3, \text{EqvChain}$$

$$\vdash (f_{1,1}; \dots; f_{1,h}); \dots; (f_{n,1}; \dots; f_{n,l_n}) \equiv f_{1,1}; \dots; f_{1,h}; \dots; f_{n,1}; \dots; f_{n,l_n}$$

ChopGroupGroupMerge

where there are n groups and for each group $1 \leq i < n$, there are l_i formulas. Proof is by induction on n .

Proof for $n = 1$:

$$1 \vdash f_{1,1}; \dots; f_{1,h} \equiv f_{1,1}; \dots; f_{1,h} \quad \text{Prop}$$

Proof for $n > 1$:

$$1 \vdash (f_{2,1}; \dots; f_{2,l_2}); \dots; (f_{n,1}; \dots; f_{n,l_n}) \equiv f_{2,1}; \dots; f_{2,l_2}; \dots; f_{n,1}; \dots; f_{n,l_n}$$

induction hypothesis

$$2 \vdash (f_{1,1}; \dots; f_{1,h}); (f_{2,1}; \dots; f_{2,l_2}); \dots; (f_{n,1}; \dots; f_{n,l_n}) \equiv (f_{1,1}; \dots; f_{1,h}); (f_{2,1}; \dots; f_{2,l_2}; \dots; f_{n,1}; \dots; f_{n,l_n})$$

1, LeftChopEqvChop

$$3 \vdash (f_{1,1}; \dots; f_{1,h}); (f_{2,1}; \dots; f_{2,l_2}; \dots; f_{n,1}; \dots; f_{n,l_n}) \equiv f_{1,1}; \dots; f_{1,h}; f_{2,1}; \dots; f_{2,l_2}; \dots; f_{n,1}; \dots; f_{n,l_n}$$

ChopGroupMerge

$$4 \vdash (f_{1,1}; \dots; f_{1,h}); (f_{2,1}; \dots; f_{2,l_2}); \dots; (f_{n,1}; f_{n,2}; \dots; f_{n,l_n}) \equiv f_{1,1}; \dots; f_{1,h}; f_{2,1}; \dots; f_{2,l_2}; \dots; f_{n,1}; \dots; f_{n,l_n}$$

2, 3, EqvChain

$$\vdash f; f; \dots; f \supset f^*$$

ChopGroupImpCS

The proof is by induction on the number of <chops>.

Proof when no <chops>:

$$1 \vdash f \supset f^* \quad \text{ImpCS}$$

Proof for at n occurrences of <chops> where $n \geq 1$:

$$1 \vdash f; \dots; f \supset f^* \quad \text{induction for } n-1 \text{ <chops>}$$

$$2 \vdash f; f; \dots; f \supset f; f^* \quad 1, \text{LeftChopImpChop}$$

$$3 \vdash f; f^* \supset f^* \quad \text{ChopCSImpCS}$$

$$4 \vdash f; f; \dots; f \supset f^* \quad 2, 3, \text{ImpChain}$$

$$\vdash f_1 \supset g, \dots, \vdash f_n \supset g \Rightarrow \vdash (f_1; \dots; f_n) \supset g^*$$

MultChopImpCS

Proof:

$$1 \vdash f_i \supset g, \quad \text{for } 1 \leq i \leq n \quad \text{given}$$

$$2 \vdash f_1; \dots; f_n \supset g; \dots; g \quad 1, \text{MultChopImpChop}$$

$$3 \vdash g; \dots; g \supset g^* \quad \text{ChopGroupImpCS}$$

$$4 \vdash f_1; \dots; f_n \supset g^* \quad 2, 3, \text{ImpChain}$$

$$\begin{aligned} &\vdash w \wedge f \supset g ; (w_1 \wedge f_1), \quad \vdash w_1 \wedge f_1 \supset g_1 ; (w_2 \wedge f_2) \\ &\Rightarrow \vdash w \wedge f \supset g ; g_1 ; (w_2 \wedge f_2) \end{aligned}$$

NestedChopImpChop

Proof:

$$\begin{array}{ll} 1 \vdash w \wedge f \supset g ; (w_1 \wedge f_1) & \text{given} \\ 2 \vdash w_1 \wedge f_1 \supset g_1 ; (w_2 \wedge f_2) & \text{given} \\ 3 \vdash g ; (w_1 \wedge f_1) \supset g ; g_1 ; (w_2 \wedge f_2) & 2, \text{RightChopImpChop} \\ 4 \vdash w \wedge f \supset g ; g_1 ; (w_2 \wedge f_2) & 1, 3, \text{ImpChain} \end{array}$$

$$\begin{aligned} &\vdash w_1 \wedge f_1 \supset g_1 ; (w_2 \wedge f_2), \dots, \vdash w_{n-1} \wedge f_{n-1} \supset g_{n-1} ; (w_n \wedge f_n) \\ &\Rightarrow \vdash w_1 \wedge f_1 \supset g_1 ; \dots ; g_{n-1} ; (w_n \wedge f_n) \end{aligned}$$

MultNestedChopImpChop

The proof is by induction on n .

Proof for $n = 1$:

$$1 \vdash w_1 \wedge f_1 \supset w_1 \wedge f_1 \quad \text{Prop}$$

Proof for $n > 1$:

$$\begin{array}{ll} 1 \vdash w_1 \wedge f_1 \supset g_1 ; (w_2 \wedge f_2) & \text{given} \\ 2 \vdash w_i \wedge f_i \supset g_i ; (w_{i+1} \wedge f_{i+1}), \quad \text{for each } i: 1 < i < n & \text{given} \\ 3 \vdash w_2 \wedge f_2 \supset g_2 ; \dots ; g_{n-1} ; (w_n \wedge f_n) & 2, \text{induction hypothesis} \\ 4 \vdash w_1 \wedge f_1 \supset g_1 ; g_2 ; \dots ; g_{n-1} ; (w_n \wedge f_n) & 1, 3, \text{NestedChopImpChop} \end{array}$$

Part II

JANCL proofs

Proofs taken from

Ben Moszkowski. “A Hierarchical Completeness Proof for Propositional Interval Temporal Logic with Finite Time”. In: *Journal of Applied Non-Classical Logics* 14.1–2 (2004), pp. 55–104. [url](#).

12 Propositional Proofs

$$\vdash f_1 \supset f_2, \dots, \vdash f_{n-1} \supset f_n \Rightarrow \vdash f_1 \supset f_n$$

ImpChain

$$\vdash f_1 \equiv f_2, \dots, \vdash f_{n-1} \equiv f_n \Rightarrow \vdash f_1 \equiv f_n$$

EqvChain

$$\vdash f_1, \vdash f_2, \dots, \vdash f_n \Rightarrow \vdash g,$$

Prop

where the formula $f_1 \wedge f_2 \wedge \dots \wedge f_n \supset g$

is a substitution instance of a propositional tautology

13 PITL Axiom System

We now present an axiom system for PITL. Our experience in rigorously developing hundreds of proofs has helped us refine the axioms and convinced us of their utility for a wide range of purposes.

Definition 1 (Tautology) *A tautology is any formula which is a substitution instance of some valid nonmodal propositional formula.*

For example, any PITL formula of the form $\Box f \vee \Diamond g \supset \Diamond g$ is a tautology since it is a substitution instance of the valid nonmodal formula $h_0 \vee h_1 \supset h_1$. It is not hard to show that all tautologies are themselves valid. Intuitively, a formula is a tautology if it does not require any modal reasoning to justify its truth.

13.1 Axioms and Inference Rules for PITL

Our PITL axiom system is given in Table 1. Recall that the symbol \supset is the logical operator *implication* used in formulas. In contrast, the metalogical symbol \Rightarrow denotes the ability to infer a new theorem from other previously deduced ones. The axiom system mainly deals with *chop*, and skip and operators derived from them. Only one axiom is needed for *chop-star*.

The axiom system contains some of the propositional axioms suggested by Rosner and Pnueli but also includes our own axioms and inference rule for the operators \Box and *chop-star*. These assist in deducing theorems and derived inference rules for compositional reasoning. The Axiom **Taut** permits using properties of conventional nonmodal logic without proof (recall Definition 1 concerning tautologies). It is possible to omit it and achieve the same results by means of a few “lower-level” axioms and inference rules dealing primarily with nonmodal reasoning.

\vdash All PITL tautologies	Taut
$\vdash(f ; g) ; h \equiv f ; (g ; h)$	ChopAssoc
$\vdash(f_0 \vee f_1) ; g \supset (f_0 ; g) \vee (f_1 ; g)$	OrChopImp
$\vdash f ; (g_0 \vee g_1) \supset (f ; g_0) \vee (f ; g_1)$	ChopOrImp
$\vdash \text{empty} ; f \equiv f$	EmptyChop
$\vdash f ; \text{empty} \equiv f$	ChopEmpty
$\vdash w \supset \Box w$	StatImpBi
$\vdash \Box(f_0 \supset f_1) \wedge \Box(g_0 \supset g_1) \supset (f_0 ; g_0) \supset (f_1 ; g_1)$	BiBoxChopImpChop
$\vdash \bigcirc f \supset \textcircled{w} f$	NextImpWeakNext
$\vdash f \wedge \Box(f \supset \textcircled{w} f) \supset \Box f$	BoxInduct
$\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}) ; f^*$	ChopStarEqv
$\vdash f \supset g, \vdash f \Rightarrow \vdash g$	MP
$\vdash f \Rightarrow \vdash \Box f$	BoxGen
$\vdash f \Rightarrow \vdash \Box \Box f$	BiGen

Table 1: PITL axiom system

The axiom system gives nearly equal treatment to initial and terminal subintervals. For example, the Inference Rules **BiGen** and **BoxGen** respectively provide a means to obtain new theorems by embedding previously deduced PITL theorems in \Box and \Box . This is exceedingly important for the kinds of proofs we do since we naturally move formulas in and out of the left side of chop in many situations. The later embedding of the FL axiom system in the PITL axiom system and the reduction of PITL completeness to FL completeness both involve a lot of this kind of reasoning. The proof of the PITL Replacement Theorem is also a good example of how the analysis of the left side of chop is relevant. We additionally believe that axioms and inference rules concerning \Box make the axiom system easier to understand since much of it consists simply of duals in this sense. In contrast, most temporal logics cannot readily handle initial subintervals since the conventional operators are point-based. Even other axiom systems for ITL largely neglect initial subintervals.

A formula f which is deducible (provable) from the axioms and inferences rules is called an *PITL theorem*, denoted $\vdash f$. When doing proofs, we can observe that a PITL subset in which the only primitive temporal operator is chop and one side is always some fixed formula obeys the rules of the conventional normal modal system K . We now give two sample theorems and their proofs. The justification **Prop** in some steps refers to conventional propositional reasoning which can involve implicit uses of Axiom **Taut** and/or modus ponens **MP**.

$$\vdash \Box(f \supset g) \supset \Diamond f \supset \Diamond g \quad \text{BilmpDilmpDiSample}$$

Proof:

1	$\text{true} \supset \text{true}$	Prop
2	$\Box(\text{true} \supset \text{true})$	1, BoxGen
3	$\Box(f \supset g) \wedge \Box(\text{true} \supset \text{true})$ $\supset (f ; \text{true}) \supset (g ; \text{true})$	BiBoxChopImpChop
4	$\Box(f \supset g) \supset (f ; \text{true}) \supset (g ; \text{true})$	2, 3, Prop
5	$\Box(f \supset g) \supset \Diamond f \supset \Diamond g$	4, def. of \Diamond

The following instance of Axiom **StateImpBi** illustrates why it is not subsumed by Inference Rule **BiGen**:

$$\vdash \neg Q \supset \Box \neg Q$$

Here Q is a propositional variable. We cannot use **BiGen** since $\neg Q$ is not a theorem.

14 Deduction of PTL Axioms from the FL Axiom System

$\vdash \Box(f \supset g) \supset \Box f \supset \Box g$	A1
$\vdash \bigcirc f \supset \textcircled{w} f$	A2
$\vdash \bigcirc(f \supset g) \supset \bigcirc F \supset \bigcirc g$	A3
$\vdash \Box f \supset f \wedge \textcircled{w} \Box f$	A4
$\vdash \Box(f \supset \textcircled{w} f) \supset f \supset \Box f$	A5
$f \text{ is a tautology} \Rightarrow \vdash f$	R1
$\vdash f \supset g, \vdash f \Rightarrow \vdash g$	R2
$\vdash f \Rightarrow \vdash \Box f$	R3

Table 2: Modified version of Pnueli's complete axiom system

This appendix contains various FL theorems and their deductions. These include ones corresponding to some of the PTL axioms in Table 2. Most of the PTL axioms and inference rules have identical or nearly identical versions in the FL axiom system in Table 3. The three exceptions are Axioms **A1**, **A3** and **A4**. We will look at each of them in turn as FL theorems **FBoxImpDist**, **FNextImpDist** and **FBoxImpNowAndWeakNext**, respectively. The trickiest is Axiom **A1**. The symbol \vdash as used here always refers to \vdash_{FL} . None of the FE formulas occurring in the proofs contain variables and therefore the proofs also ensure well-formed FLV theorems and derived inference rules for any V .

\vdash All FL tautologies	FLTaut
$\vdash \bigcirc X \equiv \langle \text{skip} \rangle f$	FL2
$\vdash \diamond f \equiv \langle \text{skip}^* \rangle f$	FL3
$\vdash \langle w? \rangle f \equiv w \wedge f$	FL4
$\vdash \langle E_0 \vee E_1 \rangle f \equiv \langle E_0 \rangle f \vee \langle E_1 \rangle f$	FL5
$\vdash \langle E_0 ; E_1 \rangle f \equiv \langle E_0 \rangle \langle E_1 \rangle f$	FL6
$\vdash \langle E \rangle (f \vee g) \supset \langle E \rangle f \vee \langle E \rangle g$	FL7
$\vdash \langle E^* \rangle f \equiv f \vee \langle E ; E^* \rangle f$	FL8
$\vdash \Box (f \supset g) \supset \langle E \rangle f \supset \langle E \rangle g$	FL9
$\vdash \bigcirc f \supset \mathbb{W}f$	FL10
$\vdash \Box (f \supset \mathbb{W}f) \wedge f \supset \Box f$	FL11
$\vdash \diamond \text{empty}$	FL12
$\vdash f \supset g, \vdash f \Rightarrow \vdash g$	FLMP
$\vdash f \Rightarrow \vdash \Box f$	FLBoxGen
$\vdash \langle E_0 \rangle \text{empty} \supset \langle E_0 \rangle \text{empty} \Rightarrow \vdash \langle E_0 \rangle f \supset \langle E_1 \rangle f$	FLnf3
$\vdash (\text{more} \wedge \langle E_0 \rangle \text{empty}) \langle E_1 \rangle \text{empty} \Rightarrow \vdash \langle E_0^* \rangle f \supset \langle E_1^* \rangle f$	FLnf4

Table 3: Axiom system for FL

$\vdash \Box (f \supset g) \supset \diamond f \supset \diamond g$	FBoxImpDiamondImpDiamond
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Proof:

- 1 $\Box (f \supset g) \supset \langle \text{skip}^* \rangle f \supset \langle \text{skip}^* \rangle g$ **FL9**
- 2 $\diamond f \equiv \langle \text{skip}^* \rangle f$ **FL3**
- 3 $\diamond g \equiv \langle \text{skip}^* \rangle g$ **FL3**
- 4 $\Box (f \supset g) \supset \diamond f \supset \diamond g$ 2, 3, **Prop**

The following slightly obscure theorem is used in the proof of **FBoxImpDist**:

$\vdash \Box (\neg g \supset \neg f) \supset \Box f \supset \Box g$	FBoxContraPosImpDist
---	----------------------

Proof:

- 1 $\Box (\neg g \supset \neg f) \supset \diamond \neg g \supset \diamond \neg f$ **FBoxImpDiamondImpDiamond**
- 2 $\Box (\neg g \supset \neg f) \supset \neg \diamond \neg f \supset \neg \diamond \neg g$ 1, **Prop**
- 3 $\Box (\neg g \supset \neg f) \supset \Box f \supset \Box g$ 2, def. of \Box

Below is the proof of PTL Axiom **A1** as FL theorem **FBoxImpDist**. In the final step, **ImpChain** stands for a chain of implications.

$$\vdash \Box(X \supset g) \supset \Box f \supset \Box g \quad \text{FBoxImpDist}$$

Proof:

- | | | |
|---|--|-----------------------------|
| 1 | $(f \supset g) \supset (\neg g \supset \neg f)$ | Prop |
| 2 | $\neg(\neg g \supset \neg f) \supset \neg(f \supset g)$ | 1, Prop |
| 3 | $\Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$ | 2, FLBoxGen |
| 4 | $\Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$
$\supset \Box(f \supset g) \supset \Box(\neg g \supset \neg f)$ | FBoxContraPosImpDist |
| 5 | $\Box(f \supset g) \supset \Box(\neg g \supset \neg f)$ | 3, 4, FLMP |
| 6 | $\Box(\neg g \supset \neg f) \supset \Box f \supset \Box g$ | FBoxContraPosImpDist |
| 7 | $\Box(X \supset g) \supset \Box f \supset \Box g$ | 5, 6, ImpChain |
-

$$\vdash \bigcirc \neg f \supset \neg \bigcirc f \quad \text{FNextNotImpNotNext}$$

Proof:

- | | | |
|---|---|--|
| 1 | $\bigcirc f \supset \textcircled{w} f$ | FL10 |
| 2 | $\bigcirc f \supset \neg \bigcirc \neg f$ | 1, def. of \textcircled{w} |
| 3 | $\bigcirc \neg f \supset \neg \bigcirc f$ | 2, Prop |

Here is a proof of PTL Axiom **A3**:

$$\vdash \bigcirc(f \supset g) \supset \bigcirc f \supset \bigcirc g \quad \text{FNextImpDist}$$

Proof:

- | | | |
|---|---|--|
| 1 | $\langle \text{skip} \rangle (\neg f \vee g) \supset (\langle \text{skip} \rangle \neg f) \vee (\langle \text{skip} \rangle g)$ | FL7 |
| 2 | $\bigcirc(\neg f \vee g) \equiv \langle \text{skip} \rangle (\neg f \vee g)$ | FL2 |
| 3 | $\bigcirc \neg f \equiv \langle \text{skip} \rangle \neg f$ | FL2 |
| 4 | $\bigcirc g \equiv \langle \text{skip} \rangle g$ | FL2 |
| 5 | $\bigcirc(\neg f \vee g) \supset \bigcirc \neg f \vee \bigcirc g$ | 1-4, Prop |
| 6 | $\bigcirc \neg f \supset \neg \bigcirc f$ | FNextNotImpNotNext |
| 7 | $\bigcirc(\neg f \vee g) \supset \neg \bigcirc f \vee \bigcirc g$ | 5, 6, Prop |
| 8 | $\bigcirc(f \supset g) \supset \bigcirc f \supset \bigcirc g$ | 7, def. of \supset |

The remaining proofs are for ultimately deducing PTL Axiom **A4** as FL theorem **FBoxImpNowAndWeakNext**. The following derived rule **FRightSkipChopImpSkipChopRule** can be readily generalised to allow some arbitrary FE formula in place of skip. In addition, a version can be proven which uses \equiv instead of \supset .

$$\vdash f \supset g \Rightarrow \vdash \langle \text{skip} \rangle f \supset \langle \text{skip} \rangle g$$

FRightSkipChopImpSkipChopRule

Proof:

1	$f \supset g$	Assump
2	$\Box(f \supset g)$	FLBoxGen
3	$\Box(f \supset g) \supset \langle \text{skip} \rangle f \supset \langle \text{skip} \rangle g$	FL9
4	$\langle \text{skip} \rangle f \supset \langle \text{skip} \rangle g$	2, 3, MP

$$\vdash f \supset g \Rightarrow \circ f \supset \circ g$$

FNextImpNextRule

Proof:

1	$f \supset g$	Assump
2	$\langle \text{skip} \rangle f \supset \langle \text{skip} \rangle g$	1, FRightSkipChopImpSkipChopRule
3	$\circ f \equiv \langle \text{skip} \rangle f$	FL2
4	$\circ g \equiv \langle \text{skip} \rangle g$	FL2
5	$\circ f \supset \circ g$	3, 4, Prop

$$\vdash f \equiv g \Rightarrow \circ f \equiv \circ g$$

FNextEqvNextRule

Proof:

1	$f \equiv g$	Assump
2	$f \supset g$	1, Prop
3	$\circ f \supset \circ g$	2, FNextImpNextRule
4	$g \supset f$	1, Prop
5	$\circ g \supset \circ f$	4, FNextImpNextRule
6	$\circ f \equiv \circ g$	3, 5, Prop

$$\vdash \diamond f \equiv f \vee \circ \diamond f$$

FDiamondEqvNowOrNextDiamond

Proof:

1	$\diamond f \equiv \langle \text{skip}^* \rangle f$	FL3
2	$\langle \text{skip}^* \rangle f \equiv f \vee \langle \text{skip} ; \text{skip}^* \rangle f$	FL8
3	$\langle \text{skip} ; \text{skip}^* \rangle f \equiv \langle \text{skip} \rangle \langle \text{skip}^* \rangle f$	FL6
4	$\circ \langle \text{skip}^* \rangle f \equiv \langle \text{skip} \rangle \langle \text{skip}^* \rangle f$	FL2
5	$\circ \diamond f \equiv \circ \langle \text{skip}^* \rangle X$	1, FNextEqvNextRule
6	$\diamond f \equiv f \vee \circ \diamond f$	1–5, Prop

$$\vdash f \supset \diamond f$$

FNowImpDiamond

Proof:

- 1 $\diamond f \equiv f \vee \bigcirc \diamond f$ **FDiamondEqvNowOrNextDiamond**
 - 2 $f \supset \diamond f$ 1, **Prop**
-

$$\vdash \bigcirc \diamond f \supset \diamond f$$

FNextDiamondImpDiamond

Proof:

- 1 $\diamond f \equiv f \vee \bigcirc \diamond f$ **FDiamondEqvNowOrNextDiamond**
 - 2 $\bigcirc \diamond f \supset \diamond f$ 1, **Prop**
-

$$\vdash \Box f \supset f$$

BoxImpNow

Proof:

- 1 $\neg f \supset \diamond \neg f$ **FNowImpDiamond**
 - 2 $\neg \diamond \neg f \supset f$ 1, **Prop**
 - 3 $\Box f \supset f$ 2, def. of \Box
-

$$\vdash \Box f \supset \textcircled{w}\Box f$$

FBoxImpWeakNextBox

Proof:

- 1 $\neg \neg \diamond \neg f \supset \diamond \neg f$ **Prop**
- 2 $\bigcirc \neg \neg \diamond \neg f \supset \bigcirc \diamond \neg f$ 1, **FNextImpNextRule**
- 3 $\bigcirc \diamond \neg f \supset \diamond \neg f$ **FNextDiamondImpDiamond**
- 4 $\bigcirc \neg \neg \diamond \neg f \supset \diamond \neg f$ 2, 3, **ImpChain**
- 5 $\bigcirc \neg \Box f \supset \diamond \neg f$ 4, def. of \Box
- 6 $\neg \diamond \neg f \supset \neg \bigcirc \neg \Box f$ 5, **Prop**
- 7 $\Box f \supset \textcircled{w}\Box f$ 6, def. of \Box, \textcircled{w}

Below is a proof of PTL Axiom **A4**:

$$\vdash \Box f \supset f \wedge \textcircled{w}\Box f$$

FBoxImpNowAndWeakNext

Proof:

- 1 $\Box f \supset f$ **BoxImpNow**
- 2 $\Box f \supset \textcircled{W}\Box f$ **FBoxImpWeakNextBox**
- 3 $\Box f \supset f \wedge \textcircled{W}\Box f$ 1, 2, **Prop**

Part III

LMCS proofs

Proofs taken from

Ben C. Moszkowski. "A Complete Axiom System for Propositional Interval Temporal Logic with Infinite Time". In: Logical Methods in Computer Science Journal 8.3 (2012). [url](#).

15 Axiom system for PITL with finite and infinite time

\vdash Substitution instances of valid PTL formulas	VPTL
$\vdash (f \frown g) \frown h \equiv f \frown (g \frown h)$	ChopAssoc
$\vdash (f_0 \vee f_1) \frown g \supset (f_0 \frown g) \vee (f_1 \frown g)$	OrChopImp
$\vdash f \frown (g_0 \vee g_1) \supset (f \frown g_0) \vee (f \frown g_1)$	ChopOrImp
$\vdash \text{empty} \frown f \equiv f$	EmptyChop
$\vdash \text{finite} \supset (f \frown \text{empty} \equiv f)$	FiniteImpChopEmpty
$\vdash w \supset \Box w$	StateImpBf
$\vdash \Box(f_0 \supset f_1) \wedge \Box(g_0 \supset g_1) \supset (f_0 \frown g_0) \supset (f_1 \frown g_1)$	BfAndBoxImpChopImpChop
$\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}) \frown f^*$	SChopStarEqv
$\vdash f \wedge \Box(f \supset (g \wedge \text{more}) \frown f) \supset g^\omega$	ChopOmegaInduct
$\vdash f \supset g, \vdash f \Rightarrow \vdash g$	MP
$\vdash \text{finite} \supset f \Rightarrow \vdash \Box f$	BfFGen
$\vdash f \Rightarrow \vdash \Box f$	BoxGen
$\vdash \Box((\text{fin } P) \equiv g) \supset f \Rightarrow \vdash f$	BfAux

In **BfAux**, propositional variable P must not occur in f and g .

$\vdash f_1 \supset f_2, \dots, \vdash f_{n-1} \supset f_n \Rightarrow \vdash f_1 \supset f_n$	ImpChain
$\vdash f_1 \equiv f_2, \dots, \vdash f_{n-1} \equiv f_n \Rightarrow \vdash f_1 \equiv f_n$	EqvChain
$\vdash f_1, \vdash f_2, \dots, \vdash f_n \Rightarrow \vdash g,$ where the formula $f_1 \wedge f_2 \wedge \dots \wedge f_n \supset g$ is a substitution instance of a propositional tautology	Prop

\vdash Substitution instances of conventional (nonmodal) tautologies	Taut
$\vdash (f \frown g) \frown h \equiv f \frown (g \frown h)$	FChopAssoc
$\vdash (f_0 \vee f_1) \frown g \supset (f_0 \frown g) \vee (f_1 \frown g)$	FOrChopImp
$\vdash f \frown (g_0 \vee g_1) \supset (f \frown g_0) \vee (f \frown g_1)$	FChopOrImp
$\vdash \text{empty} \frown f \equiv f$	FEmptyChop
$\vdash f \frown \text{empty} \equiv f$	FChopEmpty
$\vdash w \supset \boxplus w$	FStateImpBf
$\vdash \boxplus(f_0 \supset f_1) \wedge \Box(g_0 \supset g_1) \supset (f_0 \frown g_0) \supset (f_1 \frown g_1)$	FBfAndBoxImpChopImpChop
$\vdash f^* \equiv \text{empty} \vee (f \wedge \text{more}) \frown f^*$	FChopStarEqv
$\vdash \bigcirc f \supset \textcircled{w} f$	FNextImpWeakNext
$\vdash f \wedge \Box(f \supset \textcircled{w} f) \supset \Box f$	FBoxInduct
$\vdash f \supset g, \vdash f \Rightarrow \vdash g$	FMP
$\vdash f \Rightarrow \vdash \boxplus f$	FBfGen
$\vdash f \Rightarrow \vdash \Box f$	FBoxGen

Note: $\textcircled{w} f \triangleq \neg \bigcirc \neg f$ (Weak next)

Table 4: Axiom system for PITL with just finite time

16 Axiom system for PITL with finite time

17 Some PITL theorems and Their Proofs

This appendix gives a representative set of PITL theorems and derived inference rules together with their proofs. Many are used either directly or indirectly in the completeness proof for PITL with both finite and infinite time. We have partially organised the material, particularly in Section 17.2, along the lines of some standard modal logic systems.

The PITL theorems and derived rules have a shared index sequence (e.g., **BfChopImpChop–BoxChopEqvChop** are followed by **BfGen** rather than **DR1**). We believe that this convention simplifies locating material in this appendix.

Proof steps can refer to axioms, inference rules, previously deduced theorems, derived inference rules and also the following:

- Assumptions which are regarded as being previously deduced.
- **Prop**: Conventional nonmodal propositional reasoning (by restricted application of Axiom **VPTL**) and Modus Ponens.
- **ImpChain**: A chain of implications.
- **EqvChain**: A chain of equivalences.

- In principle, **ImpChain** and **EqvChain** are subsumed by **Prop** but are used here to make the reasoning more explicit.

Our assumption of axiomatic completeness for PITL with just finite time permits any valid implication of the form $\text{finite} \supset f$.

PITLF

17.1 Some Basic Properties of Chop

We now consider deducing various simple properties of chop and the associated operators \diamond , \boxplus , \diamond and \square which have a wide range of uses.

$$\vdash \boxplus(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g) \quad \text{BfChopImpChop}$$

1	$g \supset g$	Prop
2	$\square(g \supset g)$	1, BoxGen
3	$\boxplus(f \supset f_1) \wedge \square(g \supset g) \supset (f \frown g) \supset (f_1 \frown g)$	BfAndBoxImpChopImpChop
4	$\boxplus(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g)$	2, 3, Prop

$$\vdash \square(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1) \quad \text{BoxChopImpChop}$$

1	$\text{finite} \supset (f \supset f)$	Prop
2	$\boxplus(f \supset f)$	1, BfFGen
3	$\boxplus(f \supset f) \wedge \square(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	BfAndBoxImpChopImpChop
4	$\square(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	2, 3, Prop

$$\vdash \square(g \equiv g_1) \supset (f \frown g) \equiv (f \frown g_1) \quad \text{BoxChopEqvChop}$$

1	$\square(g \equiv g_1) \equiv \square(g \supset g_1) \wedge \square(g_1 \supset g)$	VPTL
2	$\square(g \supset g_1) \supset (f \frown g) \supset (f \frown g_1)$	BoxChopImpChop
3	$\square(g_1 \supset g) \supset (f \frown g_1) \supset (f \frown g)$	BoxChopImpChop
4	$\square(g \equiv g_1) \supset (f \frown g) \equiv (f \frown g_1)$	2, 3, Prop

The following derived variant of Inference Rule **BfFGen** omits the subformula finite :

$$\vdash f \Rightarrow \vdash \Box f$$

BfGen

-
- 1 f Assump
 - 2 $\text{finite} \supset f$ 1, **Prop**
 - 3 $\Box f$ 2, **BfGen**

The derived inference rule **BfGen** can also be referred to as \Box **Gen** (analogous to the inference rule **BoxGen**).

$$\vdash f \supset f_1 \Rightarrow \vdash (f \frown g) \supset (f_1 \frown g)$$

LeftChopImpChop

-
- 1 $f \supset f_1$ Assump
 - 2 $\Box(f \supset f_1)$ 1, **BfGen**
 - 3 $\Box(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g)$ **BfChopImpChop**
 - 4 $f \frown g \supset f_1 \frown g$ 2, 3, **MP**
-

$$\vdash f \equiv f_1 \Rightarrow \vdash (f \frown g) \equiv (f_1 \frown g)$$

LeftChopEqvChop

-
- 1 $f \equiv f_1$ Assump
 - 2 $f \supset f_1$ 1, **Prop**
 - 3 $f \frown g \supset f_1 \frown g$ 2, **LeftChopImpChop**
 - 4 $f_1 \supset f$ 1, **Prop**
 - 5 $f_1 \frown g \supset f \frown g$ 4, **LeftChopImpChop**
 - 6 $f \frown g \equiv f_1 \frown g$ 3, 5, **Prop**
-

$$\vdash f \supset g \Rightarrow \vdash \Diamond f \supset \Diamond g$$

DfImpDf

-
- 1 $f \supset g$ Assump
 - 2 $f \frown \text{true} \supset g \frown \text{true}$ 1, **LeftChopImpChop**
 - 3 $\Diamond f \supset \Diamond g$ 2, def. of \Diamond
-

$$\vdash f \equiv g \Rightarrow \vdash \Diamond f \equiv \Diamond g$$

DfEqvDf

1	$f \equiv g$	Assump
2	$f \wedge \text{true} \equiv g \wedge \text{true}$	1, LeftChopEqvChop
3	$\diamond f \equiv \diamond g$	2, def. of \diamond

$$\vdash g \supset g_1 \Rightarrow \vdash (f \wedge g) \supset (f \wedge g_1)$$

RightChopImpChop

1	$g \supset g_1$	Assump
2	$\Box(g \supset g_1)$	BoxGen
3	$\Box(g \supset g_1) \supset (f \wedge g) \supset (f \wedge g_1)$	BoxChopImpChop
4	$f \wedge g \supset f \wedge g_1$	2, 3, MP

$$\vdash g \equiv g_1 \Rightarrow \vdash (f \wedge g) \equiv (f \wedge g_1)$$

RightChopEqvChop

1	$g \equiv g_1$	Assump
2	$g \supset g_1$	1, Prop
3	$f \wedge g \supset f \wedge g_1$	2, RightChopImpChop
4	$g_1 \supset g$	1, Prop
5	$f \wedge g_1 \supset f \wedge g$	4, RightChopImpChop
6	$f \wedge g \equiv f \wedge g_1$	3, 5, Prop

$$\vdash f \equiv g \Rightarrow \vdash \diamond f \equiv \diamond g$$

DiamondEqvDiamond

1	$f \equiv g$	Assump
2	$\text{true} \wedge f \equiv \text{true} \wedge g$	1, RightChopEqvChop
3	$\diamond f \equiv \diamond g$	2, def. of \diamond

$$\vdash f \equiv g \Rightarrow \vdash \Box f \equiv \Box g$$

BoxEqvBox

1	$f \equiv g$	Assump
2	$\neg f \equiv \neg g$	1, Prop
3	$\diamond \neg f \equiv \diamond \neg g$	2, DiamondEqvDiamond
4	$\neg \diamond \neg f \equiv \neg \diamond \neg g$	3, Prop
5	$\Box f \equiv \Box g$	4, def. of \Box

$$\vdash \Box f \supset g \Rightarrow \vdash \Box f \supset \Box g$$

BoxImpInferBoxImpBox

1	$\Box f \supset g$	Assump
2	$\Box(\Box f \supset g)$	1, BoxGen
3	$\Box(\Box f \supset g) \supset (\Box f \supset \Box g)$	VPTL
4	$\Box f \supset \Box g$	2, 3, MP

$\vdash (f \wedge f_1) \frown g \supset f \frown g$ AndChopA

1	$f \wedge f_1 \supset f$	Prop
2	$(f \wedge f_1) \frown g \supset f \frown g$	1, LeftChopImpChop

$\vdash (f \wedge f_1) \frown g \supset f_1 \frown g$ AndChopB

1	$f \wedge f_1 \supset f_1$	Prop
2	$(f \wedge f_1) \frown g \supset f_1 \frown g$	1, LeftChopImpChop

$\vdash (f \wedge f_1) \frown g \supset (f \frown g) \wedge (f_1 \frown g)$ AndChopImpChopAndChop

1	$(f \wedge f_1) \frown g \supset f \frown g$	AndChopA
2	$(f \wedge f_1) \frown g \supset f_1 \frown g$	AndChopB
3	$(f \wedge f_1) \frown g \supset (f \frown g) \wedge (f_1 \frown g)$	1, 2, Prop

$\vdash (f \wedge f_1) \frown g \equiv (f_1 \wedge f) \frown g$ AndChopCommute

1	$f \wedge f_1 \equiv f_1 \wedge f$	Prop
2	$(f \wedge f_1) \frown g \equiv (f_1 \wedge f) \frown g$	1, LeftChopEqvChop

$\vdash (f \vee f_1) \frown g \equiv (f \frown g) \vee (f_1 \frown g)$ OrChopEqv

The proof for \supset is immediate from axiom **OrChopImp**. Here is the proof for \subset :

1	$f \supset f \vee f_1$	Prop
2	$f \frown g \supset (f \vee f_1) \frown g$	1, LeftChopImpChop
3	$f_1 \supset f \vee f_1$	Prop
4	$f_1 \frown g \supset (f \vee f_1) \frown g$	3, LeftChopImpChop
5	$(f \frown g) \vee (f_1 \frown g) \supset (f \vee f_1) \frown g$	2, 4, Prop

$$\vdash f \frown g \supset \diamond f$$

ChopImpDf

1 $g \supset \text{true}$		Prop	
2 $f \frown g \supset f \frown \text{true}$		1, RightChopImpChop	
3 $f \frown g \supset \diamond f$		2, def. of \diamond	

$$\vdash \diamond \text{empty}$$

DfEmpty

1 $\text{empty} \frown \text{true} \equiv \text{true}$		EmptyChop	
2 $\text{empty} \frown \text{true} \supset \diamond \text{empty}$		ChopImpDf	
3 $\diamond \text{empty}$		1, 2, Prop	

$$\vdash f \frown g \supset \diamond g$$

ChopImpDiamond

1 $f \supset \text{true}$		Prop	
2 $f \frown g \supset \text{true} \frown g$		1, LeftChopImpChop	
3 $f \frown g \supset \diamond g$		2, def. of \diamond	

17.2 Some Properties of \boxplus involving the Modal System K and Axiom D

The two pairs of operators \square and \diamond and \boxplus and \boxminus obey various standard properties of modal logics. Axiom **VPTL** helps streamline reasoning involving \square and \diamond . The situation with \boxplus and \boxminus is quite different since they lack a comparable axiom. Therefore, it is especially beneficial to review some conventional modal systems which assist in organising various useful deductions involving \boxplus and \boxminus .

Table 5 summarises some relevant modal systems, various associated axioms and inference rules.

System	Axiom or inference rule	Axiom or rule name
K:	$Mf \triangleq \neg L \neg f$	M-def
plus	$\vdash L(f \supset g) \supset (Lf \supset Lg)$	K
plus	$\vdash f \Rightarrow \vdash Lf$	N
T: K plus	$\vdash Lf \supset f$	T
S4: T plus	$\vdash Lf \supset LLf$	4
KD4: K plus 4 and	$\vdash Lf \supset Mf$	D

Table 5: Some standard modal systems

Within PITL, as in PTL, the operator \square can be regarded as the conventional unary *necessity* modality L and the operator \diamond as the dual *possibility* operator M . The two operators together fulfil the requirements

of the modal system S4. We do not need to explicitly prove versions of the S4 axioms in Table 5 for \Box and \Diamond . Rather, any PITL formula which is a substitution instance of a valid S4 formula involving \Box and \Diamond can be readily deduced using the PITL proof system's Axiom **VPTL**. Similarly, inference rules based on S4 can be obtained with Axiom **VPTL**, Inference Rule **BoxGen** (which corresponds to the inference rule N of S4) and modus ponens. Moreover, the PITL proof system's Axiom **VPTL** permits using *any* PITL formula which is a substitution instance of some valid PTL formula which can also contain the PTL operator \circ . In view of all this, we do not give much further consideration to aspects of S4 with \Box and \Diamond .

In contrast to \Box , the PITL operator \boxplus does not have a comprehensive axiom analogous to **VPTL**. Therefore, we need to explicitly prove in the PITL axiom system various modal properties of \boxplus and its dual \boxminus . If only finite time is allowed, then \boxplus and \boxminus act as an S4 system. However, \boxplus with infinite time permitted does not fulfil the requirements of S4, or even those of the weaker modal system T, because Axiom T fails. Instead, \boxplus with infinite time fulfils the requirements of the modal system KD4 which is strictly weaker than S4.

Here is a list of KD4's axioms and inference rules and related PITL proofs for \boxplus :

K	$\vdash L(f \supset g) \supset (Lf \supset Lg)$	Theorem BfImpDist
N	$\vdash f \Rightarrow \vdash Lf$	Derived Inf. Rule BfGen
D	$\vdash Lf \supset Mf$	Theorem BfImpDf
4	$\vdash Lf \supset LLf$	Theorem BfImpBfBf

If only finite time is allowed, then the implication D does not need to be regarded as an explicit axiom since it can be inferred from any proof system for S4.

It is also worth noting that the related operators \boxplus and \boxminus obey the modal system S4 even when infinite time is permitted. However, we prefer to work with \boxplus and \boxminus since the use of strong chop simplifies the overall PITL completeness proof.

Conventional model logics usually take L , not M , to be primitive. When we deduce standard modal properties for \boxplus and \boxminus in our PITL axiom system, we let M , which corresponds to \boxminus , be primitive and define L to be M 's dual (i.e., $LA \triangleq \neg M\neg f$). This M -based approach goes well with the PITL axioms for chop. Chellas discusses some alternative axiomatisations of modal systems with M as the primitive although none correspond directly to ours. For the system K, we can deduce implication **LImpMImpM** below for \boxplus and \boxminus (see Theorem **BfImpDfImpDf** later on) and then obtain from it together some other reasoning the more standard axiom K just presented which only mentions L :

$$\vdash L(f \supset g) \supset (Mf \supset Mg) \qquad \text{LImpMImpM}$$

The operators \Box and \boxplus together yield a *multi-modal logic* with two necessity constructs L and L' which are commutative:

$$\vdash LL'f \equiv L'Lf$$

This corresponds to our Theorem **BfBoxEqvBoxBf** given later on.

Below are various theorems and derived inference rules about \boxplus and \boxminus for obtaining the axioms M-def (Theorem **Mdef**) and K (Theorem **BfImpDist**) found in the modal system K. The associated inference rule N was already proved above as Derived Inference Rule **BfGen**. We also prove the modal axiom D (Theorem **BfImpDf**).

In the next proof's final step, recall that **EqvChain** indicates a chain of equivalences:

$$\vdash \diamond f \equiv \neg \Box \neg f$$

Mdef

-
- | | | | |
|---|--|-------------------------------------|--|
| 1 | $f \equiv \neg \neg f$ | Prop | |
| 2 | $\diamond f \equiv \diamond \neg \neg f$ | 1, DfEqvDf | |
| 3 | $\diamond \neg \neg f \equiv \neg \neg \diamond \neg \neg f$ | Prop | |
| 4 | $\diamond \neg \neg f \equiv \neg \Box \neg f$ | 3, def. of \Box | |
| 5 | $\diamond f \equiv \neg \Box \neg f$ | 2, 4, EqvChain | |
-

$$\vdash \Box(f \supset g) \supset \diamond f \supset \diamond g$$

BfImpDfImpDf

-
- | | | | |
|---|---|---|--|
| 1 | $\Box(f \supset g) \supset (f \frown \text{true}) \supset (g \frown \text{true})$ | BfChopImpChop | |
| 2 | $\Box(f \supset g) \supset \diamond f \supset \diamond g$ | 1, def. of \diamond | |
-

$$\vdash \Box(\neg g \supset \neg f) \supset (\Box f) \supset (\Box g)$$

BfContraPosImpDist

-
- | | | | |
|---|---|-------------------------------------|--|
| 1 | $\Box(\neg g \supset \neg f) \supset (\diamond \neg g) \supset (\diamond \neg f)$ | BfImpDfImpDf | |
| 2 | $\Box(\neg g \supset \neg f) \supset (\neg \diamond \neg f) \supset (\neg \diamond \neg g)$ | 1, Prop | |
| 3 | $\Box(\neg g \supset \neg f) \supset (\Box f) \supset (\Box g)$ | 2, def. of \Box | |
-

$$\vdash \Box(f \supset g) \supset (\Box f) \supset (\Box g)$$

BfImpDist

-
- | | | | |
|---|--|---------------------------|--|
| 1 | $(f \supset g) \supset (\neg g \supset \neg f)$ | Prop | |
| 2 | $\neg(\neg g \supset \neg f) \supset \neg(f \supset g)$ | 1, Prop | |
| 3 | $\Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$ | 2, BfGen | |
| 4 | $\Box(\neg(\neg g \supset \neg f) \supset \neg(f \supset g))$
$\supset \Box(f \supset g) \supset \Box(\neg g \supset \neg f)$ | BfContraPosImpDist | |
| 5 | $\Box(f \supset g) \supset \Box(\neg g \supset \neg f)$ | 3, 4, MP | |
| 6 | $\Box(\neg g \supset \neg f) \supset (\Box f) \supset (\Box g)$ | BfContraPosImpDist | |
| 7 | $\Box(f \supset g) \supset (\Box f) \supset (\Box g)$ | 5, 6, ImpChain | |
-

$$\vdash f \supset g \Rightarrow \vdash \Box f \supset \Box g$$

BfImpBfRule

1	$f \supset g$	Assump
2	$\Box(f \supset g)$	1, BfGen
3	$\Box(f \supset g) \supset (\Box f) \supset (\Box g)$	BfImpDist
4	$\Box f \supset \Box g$	2, 3, MP

$$\vdash f \equiv g \Rightarrow \vdash \Box f \equiv \Box g$$

BfEqvBfRule

1	$f \equiv g$	Assump
2	$f \supset g$	1, Prop
3	$\Box f \supset \Box g$	2, BfImpBfRule
4	$g \supset f$	1, Prop
5	$\Box g \supset \Box f$	4, BfImpBfRule
6	$\Box f \equiv \Box g$	3, 5, Prop

$$\vdash \Box(f \wedge g) \equiv \Box f \wedge \Box g$$

BfAndEqv

1	$(f \wedge g) \supset f$	Prop
2	$\Box(f \wedge g) \supset \Box f$	1, BfImpBfRule
3	$(f \wedge g) \supset g$	Prop
4	$\Box(f \wedge g) \supset \Box g$	3, BfImpBfRule
5	$f \supset (g \supset (f \wedge g))$	Prop
6	$\Box f \supset \Box(g \supset (f \wedge g))$	5, BfImpBfRule
7	$\Box(g \supset (f \wedge g)) \supset (\Box g \supset \Box(f \wedge g))$	BfImpDist
8	$\Box f \wedge \Box g \supset \Box(f \wedge g)$	6, 7, Prop
9	$\Box(f \wedge g) \equiv \Box f \wedge \Box g$	2, 4, 8, Prop

$$\vdash \Box(f \equiv g) \equiv \Box(f \supset g) \wedge \Box(g \supset f)$$

BfEqvSplit

1	$(f \equiv g) \equiv (f \supset g) \wedge (g \supset f)$	Prop
2	$\Box(f \equiv g) \equiv \Box((f \supset g) \wedge (g \supset f))$	1, BfEqvBfRule
3	$\Box((f \supset g) \wedge (g \supset f)) \equiv \Box(f \supset g) \wedge \Box(g \supset f)$	BfAndEqv
4	$\Box(f \equiv g) \equiv \Box(f \supset g) \wedge \Box(g \supset f)$	2, 3, EqvChain

$$\vdash \Box(f \equiv f_1) \supset (f \frown g) \equiv (f_1 \frown g)$$

BfChopEqvChop

- 1 $\Box(f \equiv f_1) \equiv \Box(f \supset f_1) \wedge \Box(f_1 \supset f)$ **BfEqvSplit**
 - 2 $\Box(f \supset f_1) \supset (f \frown g) \supset (f_1 \frown g)$ **BfChopImpChop**
 - 3 $\Box(f_1 \supset f) \supset (f_1 \frown g) \supset (f \frown g)$ **BfChopImpChop**
 - 4 $\Box(f \equiv f_1) \supset (f \frown g) \equiv (f_1 \frown g)$ **1 – –3, Prop**
-

$$\vdash \Box(f \equiv g) \supset \Diamond f \equiv \Diamond g$$

BfImpDfEqvDf

- 1 $\Box(f \equiv g) \supset (f \frown \text{true}) \equiv (g \frown \text{true})$ **BfChopEqvChop**
 - 2 $\Box(f \equiv g) \supset \Diamond f \equiv \Diamond g$ **1, def. of \Diamond**
-

$$\vdash \text{finite} \supset (f \equiv g) \Rightarrow \vdash \Diamond f \equiv \Diamond g$$

FinitelmpDfEqvDfRule

- 1 $\text{finite} \supset (f \equiv g)$ **Assump**
 - 2 $\Box(f \equiv g)$ **1, BfFGen**
 - 3 $\Box(f \equiv g) \supset \Diamond f \equiv \Diamond g$ **BfImpDfEqvDf**
 - 4 $\Diamond f \equiv \Diamond g$ **2, 3, MP**
-

$$\vdash \Box f \supset \Diamond f$$

BfImpDf

- 1 $f \supset (\text{empty} \supset f)$ **Prop**
 - 2 $\Box f \supset \Box(\text{empty} \supset f)$ **1, BfImpBfRule**
 - 3 $\Box(\text{empty} \supset f) \supset (\Diamond \text{empty} \supset \Diamond f)$ **BfImpDfImpDf**
 - 4 $\Box f \supset (\Diamond \text{empty} \supset \Diamond f)$ **2, 3, ImpChain**
 - 5 $\Diamond \text{empty}$ **DfEmpty**
 - 6 $\Box f \supset \Diamond f$ **4, 5, Prop**
-

$$\vdash \Diamond(f \vee g) \equiv \Diamond f \vee \Diamond g$$

DfOrEqv

- 1 $(f \vee g) \frown \text{true} \equiv (f \frown \text{true}) \vee (g \frown \text{true})$ **OrChopEqv**
 - 2 $\Diamond(f \vee g) \equiv \Diamond f \vee \Diamond g$ **1, def. of \Diamond**
-

$$\vdash \Box f \wedge (f_1 \frown g) \supset (f \wedge f_1) \frown g$$

BfAndChopImport

- 1 $f \supset (f_1 \supset f \wedge f_1)$ **Prop**
- 2 $\Box f \supset \Box(f_1 \supset f \wedge f_1)$ **1, BfImpBfRule**
- 3 $\Box(f_1 \supset f \wedge f_1) \supset (f_1 \frown g) \supset (f \wedge f_1) \frown g$ **BfChopImpChop**
- 4 $\Box f \wedge (f_1 \frown g) \supset (f \wedge f_1) \frown g$ **2, 3, Prop**

17.3 Some Properties of Chop, \diamond and \boxplus with State Formulas

$\vdash \diamond w \equiv w$

DfState

Proof for \supset

- 1 $\neg w \supset \boxplus \neg w$ **StateImpBf**
- 2 $\neg w \supset \neg \diamond \neg \neg w$ 1, def. of \boxplus
- 3 $\diamond \neg \neg w \supset w$ 2, **Prop**
- 4 $w \supset \neg \neg w$ **Prop**
- 5 $\diamond w \supset \diamond \neg \neg w$ 4, **DfImpDf**
- 6 $\diamond w \supset w$ 3, 5, **ImpChain**

Proof for \subset

- 1 $w \supset \boxplus w$ **StateImpBf**
 - 2 $\boxplus w \supset \diamond w$ **BfImpDf**
 - 3 $w \supset \diamond w$ 1, 2, **ImpChain**
-

$\vdash \boxplus w \equiv w$

BfState

-
- 1 $\diamond \neg w \equiv \neg w$ **DfState**
 - 2 $\neg \diamond \neg w \equiv w$ 1, **Prop**
 - 3 $\boxplus w \equiv w$ 2, def. of \boxplus
-

$\vdash w \frown f \supset w$

StateChop

-
- 1 $w \frown f \supset \diamond w$ **ChopImpDf**
 - 2 $\diamond w \equiv w$ **DfState**
 - 3 $w \frown f \supset w$ 1, 2, **Prop**
-

$\vdash (w \wedge f) \frown g \supset w$

StateChopExportA

-
- 1 $w \wedge f \supset w$ **Prop**
 - 2 $(w \wedge f) \frown g \supset w \frown g$ 1, **LeftChopImpChop**
 - 3 $w \frown g \supset w$ **StateChop**
 - 4 $(w \wedge f) \frown g \supset w$ 2, 3, **ImpChain**

The following lets us move a state formula into the left side of chop:

$$\vdash w \wedge (f \frown g) \supset (w \wedge f) \frown g$$

StateAndChopImport

-
- | | | |
|---|--|------------------------|
| 1 | $w \supset \Box w$ | StateImpBf |
| 2 | $w \wedge (f \frown g) \supset \Box w \wedge (f \frown g)$ | 1, Prop |
| 3 | $\Box w \wedge (f \frown g) \supset (w \wedge f) \frown g$ | BfAndChopImport |
| 4 | $w \wedge (f \frown g) \supset (w \wedge f) \frown g$ | 2, 3, ImpChain |

We can easily combine this with theorem **StateChopExportA** to deduce the equivalence below:

$$\vdash (w \wedge f) \frown g \equiv w \wedge (f \frown g)$$

StateAndChop

-
- | | | |
|---|--|------------------------------|
| 1 | $(w \wedge f) \frown g \supset w$ | StateChopExportA |
| 2 | $(w \wedge f) \frown g \supset (w \frown g) \wedge (f \frown g)$ | AndChopImpChopAndChop |
| 3 | $(w \wedge f) \frown g \supset w \wedge (f \frown g)$ | 1, 2, Prop |
| 4 | $w \wedge (f \frown g) \supset (w \wedge f) \frown g$ | StateAndChopImport |
| 5 | $w \wedge (f \frown g) \equiv (w \wedge f) \frown g$ | 3, 4, Prop |

Below is a useful corollary of **StateAndChop** used in decomposing the left side of chop:

$$\vdash (w \wedge \text{empty}) \frown f \equiv w \wedge f$$

StateAndEmptyChop

-
- | | | |
|---|--|---------------------|
| 1 | $(w \wedge \text{empty}) \frown f \equiv w \wedge (\text{empty} \frown f)$ | StateAndChop |
| 2 | $\text{empty} \frown f \equiv f$ | EmptyChop |
| 3 | $(w \wedge \text{empty}) \frown f \equiv w \wedge f$ | 1, 2, Prop |

The following is a simple corollary of **StateAndEmptyChop**:

$$\vdash (\text{empty} \wedge w) \frown f \equiv w \wedge f$$

EmptyAndStateChop

-
- | | | |
|---|--|--------------------------|
| 1 | $(\text{empty} \wedge w) \frown f \equiv (w \wedge \text{empty}) \frown f$ | AndChopCommute |
| 2 | $(w \wedge \text{empty}) \frown f \equiv w \wedge f$ | StateAndEmptyChop |
| 3 | $(\text{empty} \wedge w) \frown f \equiv w \wedge f$ | 1, 2, EqvChain |

$$\vdash \Diamond(w \wedge f) \equiv w \wedge \Diamond f$$

StateAndDf

-
- | | | |
|---|--|-----------------------|
| 1 | $(w \wedge f) \frown \text{true} \equiv w \wedge (f \frown \text{true})$ | StateAndChop |
| 2 | $\diamond(w \wedge f) \equiv w \wedge \diamond f$ | 1, def. of \diamond |
-

$$\vdash w \supset f \Rightarrow \vdash w \supset \Box f$$

StateImpBfGen

-
- | | | |
|---|---|-------------------|
| 1 | $w \supset f$ | Assump |
| 2 | $\neg f \supset \neg w$ | 1, Prop |
| 3 | $\diamond \neg f \supset \diamond \neg w$ | 2, DfImpDf |
| 4 | $\diamond \neg w \equiv \neg w$ | DfState |
| 5 | $\diamond \neg f \supset \neg w$ | 3, 4, Prop |
| 6 | $w \supset \neg \diamond \neg f$ | 5, Prop |
| 7 | $w \supset \Box f$ | 6, def. of \Box |

The following theorem can be used to do induction over time with chop:

$$\vdash f \frown g \wedge \neg(f \frown g_1) \supset f \frown (g \wedge \neg g_1)$$

ChopAndNotChopImp

-
- | | | |
|---|---|---------------------------|
| 1 | $g \supset (g \wedge \neg g_1) \vee g_1$ | Prop |
| 2 | $f \frown g \supset f \frown (g \wedge \neg g_1) \vee f \frown g_1$ | 1, LeftChopImpChop |
| 3 | $f \frown g \wedge \neg(f \frown g_1) \supset f \frown (g \wedge \neg g_1)$ | 2, Prop |

17.4 Some Properties of \Box involving the Modal System K4

We now consider how to establish for the PITL operator \Box the axiom “4” (PITL Theorem **BfImpBfBf**) found in the modal systems *K4* and *S4*.

$$\vdash \diamond \diamond f \equiv \diamond f$$

DfDfEqvDf

-
- | | | |
|---|--|---------------------------|
| 1 | $(f \frown \text{true}) \frown \text{true} \equiv f \frown (\text{true} \frown \text{true})$ | ChopAssoc |
| 2 | $\diamond \text{true} \equiv \text{true}$ | DfState |
| 3 | $(\text{true} \frown \text{true}) \equiv \text{true}$ | 2, def. of \diamond |
| 4 | $f \frown (\text{true} \frown \text{true}) \equiv f \frown \text{true}$ | 3, LeftChopEqvChop |
| 5 | $(f \frown \text{true}) \frown \text{true} \equiv f \frown \text{true}$ | 1, 4, EqvChain |
| 6 | $\diamond \diamond f \equiv \diamond f$ | 5, def. of \diamond |
-

$$\vdash \diamond \neg f \equiv \neg \Box f$$

DfNotEqvNotBf

-
- 1 $\Box f \equiv \neg \Diamond \neg f$ def. of \Box
 - 2 $\Diamond \neg f \equiv \neg \Box f$ 1, **Prop**
-

$$\vdash \Diamond \Diamond \neg f \equiv \neg \Box \Box f$$

DfDfNotEqvNotBfBf

-
- 1 $\Diamond \neg f \equiv \neg \Box f$ **DfNotEqvNotBf**
 - 2 $\Diamond \Diamond \neg f \equiv \Diamond \neg \Box f$ 1, **DfEqvDf**
 - 3 $\Diamond \neg \Box f \equiv \neg \Box \Box f$ **DfNotEqvNotBf**
 - 4 $\Diamond \Diamond \neg f \equiv \neg \Box \Box f$ 2, 3, **EqvChain**
-

$$\vdash \Box \Box f \equiv \Box f$$

BfBfEqvBf

-
- 1 $\Diamond \Diamond \neg f \equiv \Diamond \neg f$ **DfDfEqvDf**
 - 2 $\Diamond \Diamond \neg f \equiv \neg \Box \Box f$ **DfDfNotEqvNotBfBf**
 - 3 $\neg \Box \Box f \equiv \Diamond \neg f$ 1, 2, **Prop**
 - 4 $\Diamond \neg f \equiv \neg \Box f$ **DfNotEqvNotBf**
 - 5 $\neg \Box \Box f \equiv \neg \Box f$ 3, 4, **EqvChain**
 - 6 $\Box \Box f \equiv \Box f$ 5, **Prop**
-

$$\vdash \Box f \supset \Box \Box f$$

BfImpBfBf

-
- 1 $\Box \Box f \equiv \Box f$ **BfBfEqvBf**
 - 2 $\Box f \supset \Box \Box f$ 1, **Prop**

17.5 Properties Involving the PTL Operator \bigcirc

$$\vdash (\bigcirc f) \frown g \equiv \bigcirc(f \frown g)$$

NextChop

-
- 1 $(\text{skip} \frown f) \frown g \equiv \text{skip} \frown (f \frown g)$ **ChopAssoc**
 - 2 $(\bigcirc f) \frown g \equiv \bigcirc(f \frown g)$ 1, def. of \bigcirc
-

$$\vdash (w \wedge \bigcirc f) \frown g \equiv w \wedge \bigcirc(f \frown g)$$

StateAndNextChop

-
- | | | |
|---|--|---------------------|
| 1 | $(w \wedge \bigcirc f) \frown g \equiv w \wedge ((\bigcirc f) \frown g)$ | StateAndChop |
| 2 | $(\bigcirc f) \frown g \equiv \bigcirc(f \frown g)$ | NextChop |
| 3 | $(w \wedge \bigcirc f) \frown g \equiv w \wedge \bigcirc(f \frown g)$ | 1, 2, Prop |
-

$$\vdash \diamond(w \wedge \bigcirc w') \equiv w \wedge \bigcirc w'$$

DfStateAndNextEqv

-
- | | | |
|---|---|----------------------------|
| 1 | $(w \wedge \bigcirc w') \frown \text{true} \equiv w \wedge \bigcirc(w' \frown \text{true})$ | StateAndNextChop |
| 2 | $\diamond(w \wedge \bigcirc w') \equiv w \wedge \bigcirc \diamond w'$ | 1, def. of \diamond |
| 3 | $\diamond w' \equiv w'$ | DfState |
| 4 | $\text{skip} \frown \diamond w' \equiv \text{skip} \frown w'$ | 3, RightChopEqvChop |
| 5 | $\bigcirc \diamond w' \equiv \bigcirc w'$ | 4, def. of \bigcirc |
| 6 | $\diamond(w \wedge \bigcirc w') \equiv w \wedge \bigcirc w'$ | 2, 5, Prop |

17.6 Some Properties of \boxplus Together with \square

We make use of the following analogue of Theorem **DfNotEqvNotBf** for \diamond and \square :

$$\vdash \diamond \neg f \equiv \neg \square f$$

DiamondNotEqvNotBox

-
- | | | |
|---|---|-------------|
| 1 | $\diamond \neg f \equiv \neg \square f$ | VPTL |
|---|---|-------------|
-

$$\vdash \diamond \diamond f \equiv \diamond \diamond f$$

DfDiamondEqvDiamondDf

-
- | | | |
|---|--|-----------------------|
| 1 | $(\text{true} \frown f) \frown \text{true} \equiv \text{true} \frown (f \frown \text{true})$ | ChopAssoc |
| 2 | $(\diamond f) \frown \text{true} \equiv \diamond(f \frown \text{true})$ | 1, def. of \diamond |
| 3 | $\diamond \diamond f \equiv \diamond \diamond f$ | 2, def. of \diamond |
-

$$\vdash \diamond \diamond \neg f \equiv \neg \boxplus \square f$$

DfDiamondNotEqvNotBfBox

-
- | | | |
|---|---|----------------------------|
| 1 | $\diamond \neg f \equiv \neg \square f$ | DiamondNotEqvNotBox |
| 2 | $\diamond \diamond \neg f \equiv \diamond \neg \square f$ | 1, DfEqvDf |
| 3 | $\diamond \neg \square f \equiv \neg \boxplus \square f$ | DfNotEqvNotBf |
| 4 | $\diamond \diamond \neg f \equiv \neg \boxplus \square f$ | 2, 3, EqvChain |
-

$$\vdash \diamond \diamond \neg f \equiv \neg \Box \Box f$$

DiamondDfNotEqvNotBoxBf

-
- | | | |
|---|--|-----------------------------|
| 1 | $\diamond \neg f \equiv \neg \Box f$ | DfNotEqvNotBf |
| 2 | $\diamond \diamond \neg f \equiv \diamond \neg \Box f$ | 1, DiamondEqvDiamond |
| 3 | $\diamond \neg \Box f \equiv \neg \Box \Box f$ | DiamondNotEqvNotBox |
| 4 | $\diamond \diamond \neg f \equiv \neg \Box \Box f$ | 2, 3, EqvChain |
-

$$\vdash \Box \Box f \equiv \Box \Box f$$

BfBoxEqvBoxBf

-
- | | | |
|---|--|--------------------------------|
| 1 | $\diamond \diamond \neg f \equiv \diamond \diamond \neg f$ | DfDiamondEqvDiamondDf |
| 2 | $\diamond \diamond \neg f \equiv \neg \Box \Box f$ | DfDiamondNotEqvNotBfBox |
| 3 | $\diamond \diamond \neg f \equiv \neg \Box \Box f$ | DiamondDfNotEqvNotBoxBf |
| 4 | $\Box \Box f \equiv \Box \Box f$ | 1 – 3, Prop |

17.7 Some Properties of Chop-Star

We now consider some theorems and derived rules concerning chop-star.

$$\vdash f \supset \text{more} \Rightarrow \vdash f^* \equiv \text{empty} \vee (f \frown f^*)$$

ImpMoreChopStarEqvRule

-
- | | | |
|---|--|---------------------------|
| 1 | $f \supset \text{more}$ | Assump |
| 2 | $f \wedge \text{more} \equiv f$ | 1, Prop |
| 3 | $(f \wedge \text{more}) \frown f^* \equiv f \frown f^*$ | 2, LeftChopEqvChop |
| 4 | $f^* \equiv \text{empty} \vee ((f \wedge \text{more}) \frown f^*)$ | SChopStarEqv |
| 5 | $f^* \equiv \text{empty} \vee (f \frown f^*)$ | 3, 4, Prop |
-

$$\vdash f \supset \text{more} \Rightarrow \vdash f^* \frown g \equiv g \vee (f \frown (f^* \frown g))$$

ImpMoreChopStarChopEqvRule

-
- | | | |
|---|---|----------------------------------|
| 1 | $f \supset \text{more}$ | Assump |
| 2 | $f^* \equiv \text{empty} \vee (f \frown f^*)$ | 1, ImpMoreChopStarEqvRule |
| 3 | $f^* \frown g \equiv (\text{empty} \vee (f \frown f^*)) \frown g$ | 2, LeftChopEqvChop |
| 4 | $(\text{empty} \vee (f \frown f^*)) \frown g \equiv (\text{empty} \frown g) \vee ((f \frown f^*) \frown g)$ | OrChopEqv |
| 5 | $\text{empty} \frown g \equiv g$ | EmptyChop |
| 6 | $(f \frown f^*) \frown g \equiv f \frown (f^* \frown g)$ | ChopAssoc |
| 7 | $f^* \frown g \equiv g \vee (f \frown (f^* \frown g))$ | 3 – 6, Prop |
-

$$\vdash f^* \equiv (f^* \frown \text{empty}) \vee f^\omega$$

SChopStarEqvSChopstarChopEmptyOrChopOmega

1	finite \vee \neg finite	Prop
2	finite \vee inf	1, def. of inf
3	finite $\supset (f^* \frown \text{empty}) \equiv f^*$	FiniteImpChopEmpty
4	inf $\supset f^* \equiv (f^* \wedge \text{inf})$	Prop
5	inf $\supset f^* \equiv f^\omega$	4, def. of chop-omega
6	$f^* \equiv (f^* \frown \text{empty}) \vee f^\omega$	2, 3, 5, Prop

17.8 Some Properties Involving a Reduction to PITL with Finite Time

We now present some derived inference rules which come in useful when completeness for PITL with finite time is assumed. Recall that any valid implication of the form finite $\supset f$ is allowed and that we designate such a step by using **PITLF**. PITL Theorem **BfFinStateEqvBox** below illustrates this technique.

$$\vdash \text{finite} \supset (f \supset g) \Rightarrow \vdash \Box f \supset \Box g$$

FiniteImpBfImpBfRule

1	finite $\supset (f \supset g)$	Assump
2	$\Box(f \supset g)$	1, BfFGen
3	$\Box(f \supset g) \supset (\Box f \supset \Box g)$	BfImpDist
4	$\Box f \supset \Box g$	2, 3, MP

$$\vdash \text{finite} \supset (f \equiv g) \Rightarrow \vdash \Box f \equiv \Box g$$

FiniteImpBfEqvBfRule

1	finite $\supset (f \equiv g)$	Assump
2	finite $\supset (f \supset g)$	1, Prop
3	$\Box f \supset \Box g$	2, FiniteImpBfImpBfRule
4	finite $\supset (g \supset f)$	1, Prop
5	$\Box g \supset \Box f$	4, FiniteImpBfImpBfRule
6	$\Box f \equiv \Box g$	3, 5, Prop

The next theorem's proof involves the application of the previous derived inference rule together with completeness for PITL with just finite time:

$$\vdash \Box \text{fin } w \equiv \Box w$$

BfFinStateEqvBox

1	$\Box \Box \text{fin } w \equiv \Box \text{fin } w$	BfBfEqvBf
2	$\Box \text{fin } w \equiv \Box \Box \text{fin } w$	1, Prop
3	$\text{finite} \supset ((\Box \text{fin } w) \equiv \Box w)$	PITLF
4	$\Box \Box \text{fin } w \equiv \Box \Box w$	3, FiniteImpBfEqvBfRule
5	$\Box \Box w \equiv \Box \Box w$	BfBoxEqvBoxBf
6	$\Box w \equiv w$	BfState
7	$\Box \Box w \equiv \Box w$	6, BoxEqvBox
8	$\Box \text{fin } w \equiv \Box w$	2, 4, 5, 7, EqvChain

An alternative proof of Theorem **BfFinStateEqvBox** can be given without **PITLF** by first deducing the dual equivalence $(\Diamond \Diamond(\text{empty} \wedge w)) \equiv \Diamond w$, for any state formula w .

17.9 Some Properties of Skip, Next And Until

Recall that NL^1 formulas are exactly those PTL formulas in which the only temporal operators are unnested \circ_s (e.g., $P \vee \circ \neg P$ but not $P \vee \circ \circ \neg P$). The next theorem holds for any NL^1 formula T :

$$\vdash \Diamond(\text{more} \wedge T) \equiv \text{more} \wedge T \quad \text{DfMoreAndNLoneEqvMoreAndNLone}$$

Proof 1 We use Axiom **VPTL** to re-express $\text{more} \wedge T$ as a logically equivalent disjunction $\bigvee_{1 \leq i \leq n} (w_i \wedge \circ w'_i)$ for some natural number $n \geq 1$ and n pairs of state formulas w_i and w'_i :

$$\vdash \text{more} \wedge T \equiv \bigvee_{1 \leq i \leq n} (w_i \wedge \circ w'_i) \quad \text{DfMoreAndNLoneEqvMoreAndNLone-1-eq}$$

Now by Theorem **DfStateAndNextEqv** any conjunction $w \wedge \circ w'$ is deducibly equivalent to $\Diamond(w \wedge \circ w')$. Therefore the disjunction in **DfMoreAndNLoneEqvMoreAndNLone-1-eq** can be re-expressed as $\bigvee_{1 \leq i \leq n} \Diamond(w_i \wedge \circ w'_i)$:

$$\vdash \bigvee_{1 \leq i \leq n} (w_i \wedge \circ w'_i) \equiv \bigvee_{1 \leq i \leq n} \Diamond(w_i \wedge \circ w'_i) \quad \text{DfMoreAndNLoneEqvMoreAndNLone-2-eq}$$

Then by $n - 1$ applications of Theorem **DfOrEqv** and some simple propositional reasoning, the righthand operand of this equivalence is itself deducibly equivalent to $\Diamond(\bigvee_{1 \leq i \leq n} (w_i \wedge \circ w'_i))$:

$$\vdash \bigvee_{1 \leq i \leq n} \Diamond(w_i \wedge \circ w'_i) \equiv \Diamond(\bigvee_{1 \leq i \leq n} (w_i \wedge \circ w'_i)) \quad \text{DfMoreAndNLoneEqvMoreAndNLone-3-eq}$$

The chain of the three equivalences
DfMoreAndNLoneEqvMoreAndNLone-1-eq,
DfMoreAndNLoneEqvMoreAndNLone-2-eq, and
DfMoreAndNLoneEqvMoreAndNLone-3-eq
yields the following:

$$\vdash \text{more} \wedge T \equiv \diamond \left(\bigvee_{1 \leq i \leq n} (w_i \wedge \circ w_i') \right)$$

We then apply Derived Rule **DfEqvDf** to the first equivalence **DfMoreAndNLoneEqvMoreAndNLone-1-eq**:

$$\vdash \diamond(\text{more} \wedge T) \equiv \diamond \left(\bigvee_{1 \leq i \leq n} (w_i \wedge \circ w_i') \right)$$

The last two equivalences with simple propositional reasoning yield our goal **DfMoreAndNLoneEqvMoreAndNLone**.

Here is a corollary of the previous PITL Theorem **DfMoreAndNLoneEqvMoreAndNLone** for any NL^1 formula T :

$$\vdash \boxplus(\text{more} \supset T) \equiv \text{more} \supset T \quad \text{BfMoreImpNLoneEqvMoreImpNLone}$$

1	$\boxplus(\text{more} \supset T) \equiv \neg \diamond \neg(\text{more} \supset T)$	def. of \boxplus
2	$\neg(\text{more} \supset T) \equiv \text{more} \wedge \neg T$	Prop
3	$\diamond \neg(\text{more} \supset T) \equiv \diamond(\text{more} \wedge \neg T)$	2, DfEqvDf
4	$\diamond(\text{more} \wedge \neg T) \equiv \text{more} \wedge \neg T$	DfMoreAndNLoneEqvMoreAndNLone
5	$\diamond \neg(\text{more} \supset T) \equiv \text{more} \wedge \neg T$	3, 4, EqvChain
6	$\boxplus(\text{more} \supset T) \equiv \neg(\text{more} \wedge \neg T)$	1, 5, Prop
7	$\neg(\text{more} \wedge \neg T) \equiv \text{more} \supset T$	Prop
8	$\boxplus(\text{more} \supset T) \equiv \text{more} \supset T$	6, 7, EqvChain

$$\vdash \text{more} \wedge T \supset \boxplus(\text{more} \supset T) \quad \text{MoreAndNLoneImpBfMoreImpNLone}$$

1	$\boxplus(\text{more} \supset T) \equiv \text{more} \supset T$	BfMoreImpNLoneEqvMoreImpNLone
2	$\text{more} \wedge T \supset \boxplus(\text{more} \supset T)$	1, Prop

$$\vdash \boxplus(\text{skip} \supset f) \wedge \circ g \supset (\text{skip} \wedge f) \frown g \quad \text{BfSkipImpAndNextImpAndSkipAndChop}$$

1 $\Box(\text{skip} \supset f) \wedge (\text{skip} \frown g) \supset ((\text{skip} \supset f) \wedge \text{skip}) \frown g$	BfAndChopImport
2 $(\text{skip} \supset f) \wedge \text{skip} \supset \text{skip} \wedge f$	Prop
3 $((\text{skip} \supset f) \wedge \text{skip}) \frown g \supset (\text{skip} \wedge f) \frown g$	2, LeftChopImpChop
4 $\Box(\text{skip} \supset f) \wedge (\text{skip} \frown g) \supset (\text{skip} \wedge f) \frown g$	1, 3, Prop
5 $\Box(\text{skip} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$	4, def. of \bigcirc

$\vdash \Box(\text{more} \supset f) \supset \Box(\text{skip} \supset f)$ BfMoreImplmpBfSkiImp

1 $\text{more} \supset \text{skip}$	VPTL
2 $(\text{more} \supset f) \supset (\text{skip} \supset f)$	1, Prop
3 $\Box(\text{more} \supset f) \supset \Box(\text{skip} \supset f)$	2, BfImpBfRule

$\vdash \Box(\text{more} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$ BfMoreImpAndNextImpAndSkipAndChop

1 $\Box(\text{more} \supset f) \supset \Box(\text{skip} \supset f)$	BfMoreImplmpBfSkiImp
2 $\Box(\text{skip} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$	BfSkiImpAndNextImpAndSkipAndChop
3 $\Box(\text{more} \supset f) \wedge \bigcirc g \supset (\text{skip} \wedge f) \frown g$	1, 2, Prop

$\vdash \Diamond(\text{skip} \wedge T) \equiv \text{more} \wedge T$ DfSkipAndNLoneEqvMoreAndNLone

1 $\text{finite} \supset \Diamond(\text{skip} \wedge T) \equiv (\text{more} \wedge T)$	PITLF
2 $\Diamond \Diamond(\text{skip} \wedge T) \equiv \Diamond(\text{more} \wedge T)$	1, FinitelmpDfEqvDfRule
3 $\Diamond \Diamond(\text{skip} \wedge T) \equiv \Diamond(\text{skip} \wedge T)$	DfDfEqvDf
4 $\Diamond(\text{more} \wedge T) \equiv \text{more} \wedge T$	DfMoreAndNLoneEqvMoreAndNLone
5 $\Diamond(\text{skip} \wedge T) \equiv \text{more} \wedge T$	2 – 4, Prop

$\vdash \Diamond(\text{skip} \wedge T) \supset \Box(\text{skip} \supset T)$ DfSkipAndNLoneImpBfSkiImpNLone

1 $\Diamond(\text{skip} \wedge T) \equiv \text{more} \wedge T$	DfSkipAndNLoneEqvMoreAndNLone
2 $\Diamond(\text{skip} \wedge \neg T) \equiv \text{more} \wedge \neg T$	DfSkipAndNLoneEqvMoreAndNLone
3 $\text{more} \wedge T \supset \neg(\text{more} \wedge \neg T)$	Prop
4 $\Diamond(\text{skip} \wedge T) \supset \neg \Diamond(\text{skip} \wedge \neg T)$	1 – 3, Prop
5 $\text{skip} \wedge \neg T \equiv \neg(\text{skip} \supset T)$	Prop
6 $\Diamond(\text{skip} \wedge \neg T) \equiv \Diamond \neg(\text{skip} \supset T)$	5, DfEqvDf
7 $\neg \Diamond(\text{skip} \wedge \neg T) \equiv \neg \Diamond \neg(\text{skip} \supset T)$	6, Prop
8 $\Diamond(\text{skip} \wedge T) \supset \neg \Diamond \neg(\text{skip} \supset T)$	4, 7, Prop
9 $\Diamond(\text{skip} \wedge T) \supset \Box(\text{skip} \supset T)$	8, def. of \Box

$$\vdash (\text{skip} \wedge T) \frown f \equiv T \wedge \bigcirc f$$

NLoneAndSkipChopEqvNLoneAndNext

Proof for \supset

- | | | |
|---|---|--------------------------------------|
| 1 | $(\text{skip} \wedge T) \frown f \supset \blacklozenge(\text{skip} \wedge T)$ | ChopImpDf |
| 2 | $\blacklozenge(\text{skip} \wedge T) \equiv \text{more} \wedge T$ | DfSkipAndNLoneEqvMoreAndNLone |
| 3 | $(\text{skip} \wedge T) \frown f \supset T$ | 1, 2, Prop |
| 4 | $(\text{skip} \wedge T) \frown f \supset \text{skip} \frown f$ | AndChopA |
| 5 | $(\text{skip} \wedge T) \frown f \supset \bigcirc f$ | 4, def. of \bigcirc |
| 6 | $(\text{skip} \wedge T) \frown f \supset T \wedge \bigcirc f$ | 3, 5, Prop |

Proof for \subset

- | | | |
|---|---|--|
| 1 | $\bigcirc f \supset \text{more}$ | VPTL |
| 2 | $\text{more} \wedge T \supset \boxplus(\text{more} \supset T)$ | MoreAndNLoneImpBfMoreImpNLone |
| 3 | $T \wedge \bigcirc f \supset \boxplus(\text{more} \supset T)$ | 1, 2, Prop |
| 4 | $\boxplus(\text{more} \supset T) \wedge \bigcirc f \supset (\text{skip} \wedge T) \frown f$ | BfMoreImpAndNextImpAndSkipAndChop |
| 5 | $T \wedge \bigcirc f \supset (\text{skip} \wedge T) \frown f$ | 3, 4, Prop |

$$\vdash T \text{ until } f \equiv f \vee (T \wedge \bigcirc(T \text{ until } f))$$

UntilEqv

- | | | |
|---|---|--|
| 1 | $\text{skip} \wedge T \supset \text{more}$ | VPTL |
| 2 | $(\text{skip} \wedge T)^* \frown f \equiv f \vee ((\text{skip} \wedge T) \frown ((\text{skip} \wedge T)^* \frown f))$ | 1, ImpMoreChopStarChopEqvRule |
| 3 | $T \text{ until } f \equiv f \vee ((\text{skip} \wedge T) \frown (T \text{ until } f))$ | 2, def. of until |
| 4 | $(\text{skip} \wedge T) \frown (T \text{ until } f) \equiv T \wedge \bigcirc(T \text{ until } f)$ | NLoneAndSkipChopEqvNLoneAndNext |
| 5 | $T \text{ until } f \equiv f \vee (T \wedge \bigcirc(T \text{ until } f))$ | 3 – 4, Prop |

$$\vdash T \text{ until } f \supset \blacklozenge f$$

UntillImpDiamond

- | | | |
|---|---|-----------------------|
| 1 | $(\text{skip} \wedge T)^* \frown f \supset \blacklozenge f$ | ChopImpDiamond |
| 2 | $T \text{ until } f \supset \blacklozenge f$ | 1, def. of until |

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