Interval Temporal Logic
A not so short introduction

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Website ITL course
Slides ITL course (4 slides on 1 page)

Overview (2)

This course will give a not so short introduction to Interval Temporal Logic (ITL)

ITL is a

- discrete, linear temporal logic
- for both finite and infinite intervals (sequences of states) which includes
- a basic construct for sequential composition and
- an analog of Kleene star (regular expressions)

Features of ITL (3)

- Combines temporal logic, automata and regular expressions
- Modular reasoning about time (e.g., hardware, software, multimedia)
- Flexible notation for discrete linear order
- Analytical framework for going from infinite-time behaviour to finite-time behaviour.
- Supports sequential operators found in programs, etc.

Features of ITL (4)

- Compositionality with assumptions and commitments
- Refinement to derive concrete programs from abstract specifications.
- ITL with memory and framing can embed various kinds of imperative programs, including parallel ones.
- Executable specifications and imperative subsets.
- Temporal projection between different time granularities; enables imperative constructs for interleaved processes.

ITL's Influence (5)

- ITL helped influence Prof Mike Gordon, FRS to switch from LCF to HOL in his theorem proving tools. (E.g., see his article *From LCF to HOL: A short history*).
- Hybrid systems: Duration Calculus (real and discrete time).
- ITL is used to give semantics to Verilog (EPSRC project with Oxford University).
- ITL is used in hardware/software codesign (EPSRC projects with University of Newcastle upon Tyne and Oxford University).
- ITL is used for security and trust policies (DTC-DIF projects).
- ITL used in the projects Safemos (UK) and ProCoS (EU).
- MexitI (Kent, Winnipeg) is based on ITL, Multimedia in Executable Interval Temporal Logic.

ITL's Influence (6)

- AnaTempura runtime verification tool uses executable subset of ITL.
- Influenced Verisity Ltd.'s language temporal e (part of IEEE Standard 1647). Verisity acquired by Cadence Design Systems, Inc., a major supplier of electronic design tools and services.
- Used in the KIV theorem prover (University of Augsburg, Germany) - e.g., for Statecharts, UML and medical protocols.
- ITL with framing and temporal projection extensively studied by Duan and group (Xidian University, Xi'an, China).

Part I: Propositional Logic (7)

Part II: Propositional ITL (8)

Part III: First Order Logic (9)

Part IV: First Order ITL (10)

Part I

Propositional Logic

Outline (12)

Propositional Logic (13)

Syntax of Propositional Logic:

- Boolean values: true, false
- Boolean variables: p, q, ...
- Boolean operators:

  $\wedge$ (and),
  $\vee$ (or),
  $\neg$ (not),
  $\supset$ (implication),
  $\equiv$ (equivalence)

Propositional Logic (14)

Syntax of propositional formulae in BNF:

$$f ::= \text{true} \mid p \mid \neg f \mid f_1 \wedge f_2$$

Derived Boolean operators:

$$\begin{array}{lcl} \text{false} & \widehat{=} & \neg\text{true} \\ f_1 \vee f_2 & \widehat{=} & \neg(\neg f_1 \wedge \neg f_2) \\ f_1 \supset f_2 & \widehat{=} & \neg f_1 \vee f_2 \\ f_1 \equiv f_2 & \widehat{=} & (f_1 \supset f_2) \wedge (f_2 \supset f_1) \end{array}$$

Propositional Logic Examples (15)

Example

true
$\neg \boldsymbol{p}$
true $\wedge$ $\boldsymbol{p}$
false $\vee$ $\neg \boldsymbol{p}$
false $\supset$ $\boldsymbol{p}$
$\boldsymbol{p} \equiv (\boldsymbol{p} \wedge \boldsymbol{r})$

State

A state is a mapping ***State*** from the set of propositional variables $\boldsymbol{Var^b}$ to the set of Boolean values $\boldsymbol{Bool} \mathrel{\widehat{=}} \{\mathrm{tt}, \mathrm{ff}\}$. tt is the semantic 'true' value and ff the semantic 'false' value.

$$\boldsymbol{State} : \boldsymbol{Var^b} \to \boldsymbol{Bool}$$

We will use $\sigma_0, \sigma_1, \sigma_2, \ldots$ to denote states and Σ to denote the set of all possible states.

Example

Let σ_0 be a state such that

$$\sigma_0(p) = \mathrm{tt}$$
$$\sigma_0(q) = \mathrm{ff}$$

Semantic Boolean operators (17)

Truth tables for Semantic Boolean operators:

- not :

X	not *X*
tt	ff
ff	tt

- and :

X	*Y*	*X* and *Y*
tt	tt	tt
tt	ff	ff
ff	tt	ff
ff	ff	ff

or :

X	*Y*	*X* or *Y*
tt	tt	tt
tt	ff	tt
ff	tt	tt
ff	ff	ff

- if :

X	*Y*	*X* implies *Y*
tt	tt	tt
tt	ff	ff
ff	tt	tt
ff	ff	tt

iff :

X	*Y*	*X* iff *Y*
tt	tt	tt
tt	ff	ff
ff	tt	ff
ff	ff	tt

Semantics

Let $[\![\ldots]\!]$ be the "meaning" function from ***Propositions*** $\times$ ***State*** to $\{$tt, ff$\}$ and let σ_0 be a state then

$$
\begin{array}{lcl}
[\![\text{true}]\!]_{\sigma_0} & \widehat{=} & \text{tt} \\
[\![\boldsymbol{p}]\!]_{\sigma_0} & \widehat{=} & \sigma_0(\boldsymbol{p}) \\
[\![\neg \boldsymbol{f}]\!]_{\sigma_0} & \widehat{=} & \text{not } ([\![\boldsymbol{f}]\!]_{\sigma_0}) \\
[\![\boldsymbol{f}_1 \wedge \boldsymbol{f}_2]\!]_{\sigma_0} & \widehat{=} & ([\![\boldsymbol{f}_1]\!]_{\sigma_0} \text{ and } [\![\boldsymbol{f}_2]\!]_{\sigma_0})
\end{array}
$$

Example

Let $\sigma_0(p) =$ tt and $\sigma_0(q) =$ ff.

$$
\begin{aligned}
& [\![p \vee q]\!]_{\sigma_0} \\
= \; & [\![\neg(\neg p \wedge \neg q)]\!]_{\sigma_0} \\
= \; & \text{not } ([\![\neg p \wedge \neg q]\!]_{\sigma_0}) \\
= \; & \text{not } ([\![\neg p]\!]_{\sigma_0} \text{ and } [\![\neg q]\!]_{\sigma_0}) \\
= \; & \text{not } (\text{not } ([\![p]\!]_{\sigma_0}) \text{ and not } ([\![q]\!]_{\sigma_0})) \\
= \; & \text{not } (\text{not } (\sigma_0(p)) \text{ and not } (\sigma_0(q))) \\
= \; & \text{not } (\text{not } (\text{tt}) \text{ and not } (\text{ff})) \\
= \; & \text{not } (\text{ff and tt}) \\
= \; & \text{not } (\text{ff}) \\
= \; & \text{tt}
\end{aligned}
$$

Exercises

Exercise

One can give the truth table for Boolean formulae in a similar way as the Semantic Boolean operators, i.e., for $\wedge$:

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

Give the truth table for the following Boolean formulae:

$(\neg p) \vee (q \wedge r)$
$(\neg p) \equiv (q \vee \neg r)$

Exercises

Exercise

Let $\sigma_0(p) = tt$ *and* $\sigma_0(q) = tt$.
Give the semantics of $p \equiv q$, *i.e., calculate* $[\![p \equiv q]\!]_{\sigma_0}$.

Exercise

Show that for any state σ_0 *and for propositional variables* p *and* q *the following holds* $[\![p \vee q]\!]_{\sigma_0} = ([\![p]\!]_{\sigma_0}$ *or* $[\![q]\!]_{\sigma_0})$.

Part II

Propositional Interval Temporal Logic

Outline (23)

Propositional ITL (24)

Syntax of Propositional ITL:

- Boolean values: true, false
- Boolean static variables: $\boldsymbol{p}, \boldsymbol{q}, \ldots$
- Boolean state variables: $\boldsymbol{P}, \boldsymbol{Q}, \ldots$
- Boolean operators: $\wedge, \vee, \neg, \supset, \equiv$
- Temporal operators:

skip	(skip),
;	(chop),
*	(chopstar),
$\bigcirc$	(next),
$\Box$	(always),
$\Diamond$	(sometimes), ...

Propositional ITL (25)

Syntax of Propositional ITL in BNF:

$$f ::= \text{true} \mid q \mid Q \mid \neg f \mid f_1 \wedge f_2 \mid \text{skip} \mid f_1 ; f_2 \mid f^*$$

Derived ITL operators:

$$\begin{array}{lcl} \bigcirc f & \widehat{=} & \text{skip} ; f \\ \text{inf} & \widehat{=} & \text{true} ; \text{false} \\ \text{finite} & \widehat{=} & \neg\text{inf} \\ \Diamond f & \widehat{=} & \text{finite} ; f \\ \Box f & \widehat{=} & \neg(\Diamond \neg f) \\ \ldots & & \end{array}$$

Propositional ITL Examples (26)

Example

skip

skip ; skip

$\boldsymbol{Q} \wedge \text{skip}$

$\boldsymbol{Q} \wedge (\text{skip} ; \neg \boldsymbol{Q})$

$\boldsymbol{P} \wedge (((\neg \boldsymbol{R}) \wedge \text{skip}) ; \boldsymbol{Q})$

$\boldsymbol{Q} \wedge (\bigcirc \boldsymbol{Q}) \wedge (\bigcirc\bigcirc \neg \boldsymbol{Q})$

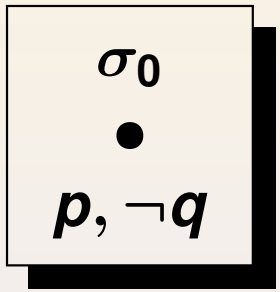
$(\Diamond \boldsymbol{P}) \wedge (\Diamond \boldsymbol{Q})$

$\Box \boldsymbol{Q}$

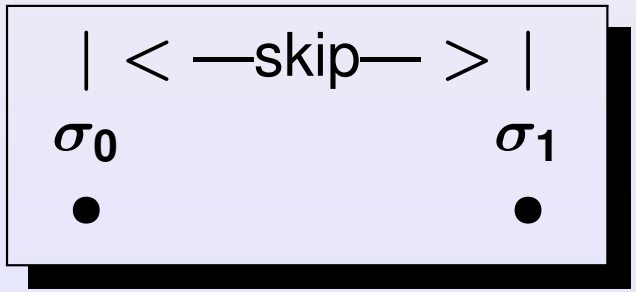
skip^*

Informal Semantics (27)

In propositional logic semantics is given wrt a state:
If $\sigma_0(p) = \text{tt}$ and $\sigma_0(q) = \text{ff}$ then $[\![p \vee q]\!]_{\sigma_0} = \text{tt}$

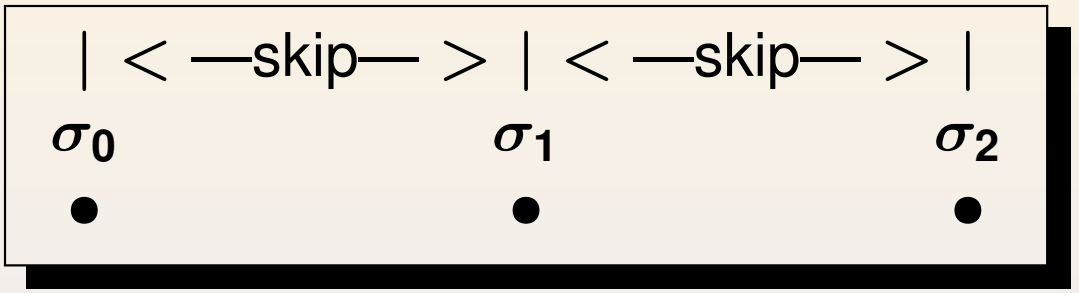


In propositional ITL semantics is given wrt a sequence of states:
Let σ_0 and σ_1 be states then $[\![\text{skip}]\!]_{\sigma_0\sigma_1} = \text{tt}$

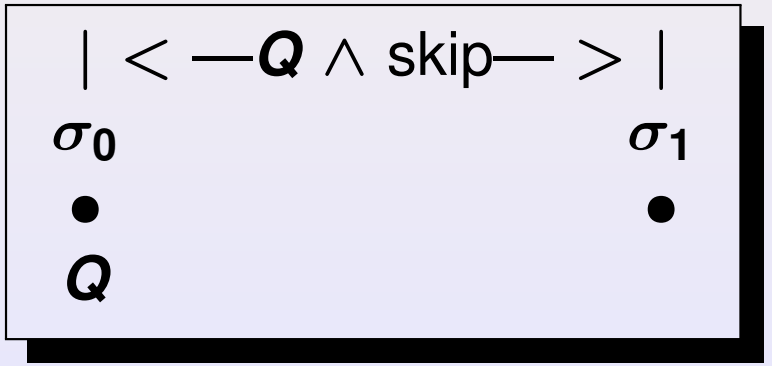


Informal Semantics (28)

Let σ_0, σ_1 and σ_2 be states then $[\![\text{skip ; skip}]\!]_{\sigma_0\sigma_1\sigma_2} = \text{tt}$

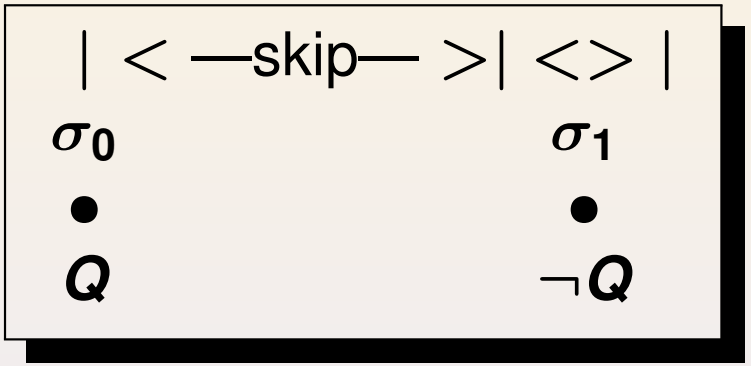


Let σ_0 and σ_1 be states and $\sigma_0(Q) = \text{tt}$ then
$[\![Q \wedge \text{skip}]\!]_{\sigma_0\sigma_1} = \text{tt}$

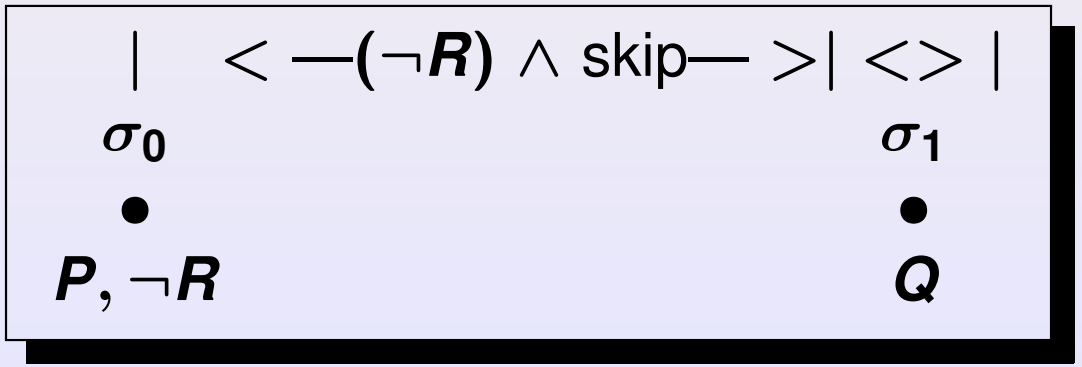


Informal Semantics (29)

Let σ_0 and σ_1 be states and $\sigma_0(Q) = \text{tt}$ and $\sigma_1(Q) = \text{ff}$ then $[\![Q \wedge (\text{skip} ; \neg Q)]\!]_{\sigma_0\sigma_1} = \text{tt}$

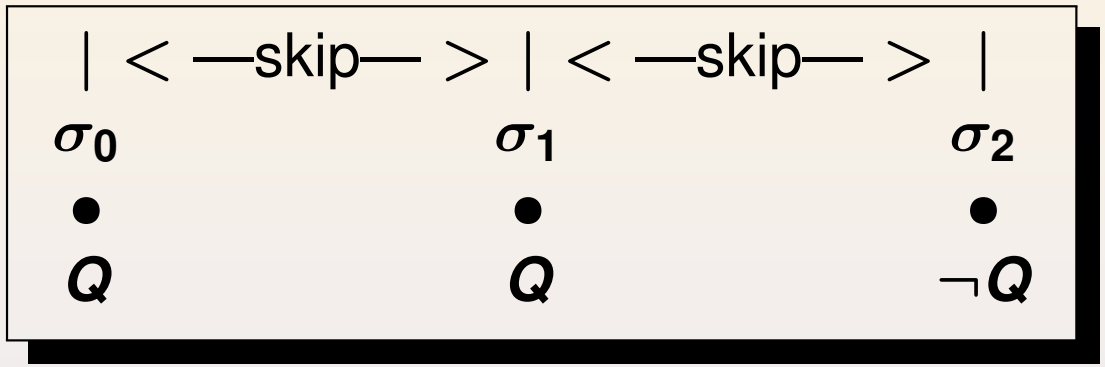


Let σ_0 and σ_1 be states and $\sigma_0(P) = \text{tt}$, $\sigma_0(R) = \text{ff}$ and $\sigma_1(Q) = \text{tt}$ then $[\![P \wedge (((\neg R) \wedge \text{skip}) ; Q)]\!]_{\sigma_0\sigma_1} = \text{tt}$

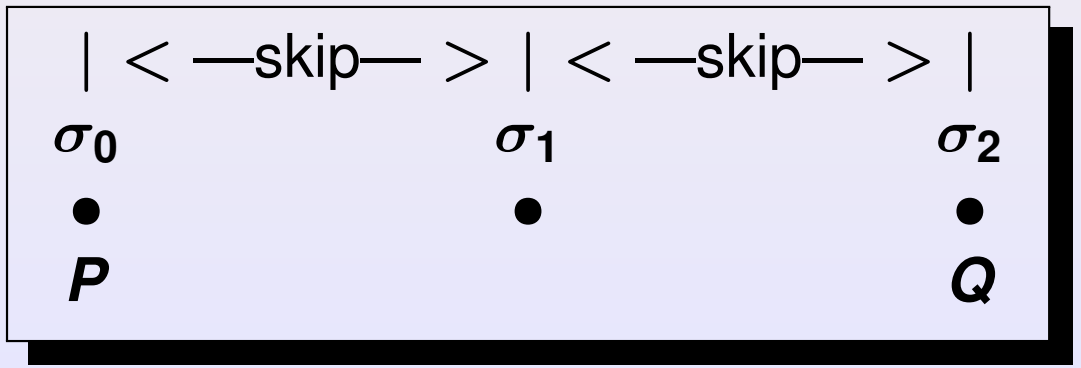


Informal Semantics (30)

Let σ_0, σ_1 and σ_2 be states and $\sigma_0(Q) = \text{tt}$, $\sigma_1(Q) = \text{tt}$ and $\sigma_2(Q) = \text{ff}$ then $[\![Q \wedge (\bigcirc Q) \wedge (\bigcirc\bigcirc \neg Q)]\!]_{\sigma_0\sigma_1\sigma_2} = \text{tt}$



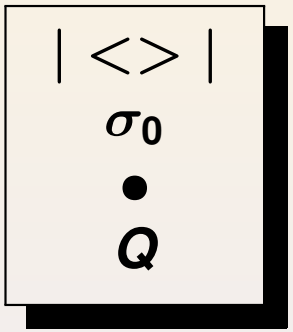
Let σ_0, σ_1 and σ_2 be states and $\sigma_0(P) = \text{tt}$ and $\sigma_2(Q) = \text{tt}$ then $[\![(\Diamond P) \wedge (\Diamond Q)]\!]_{\sigma_0\sigma_1\sigma_2} = \text{tt}$



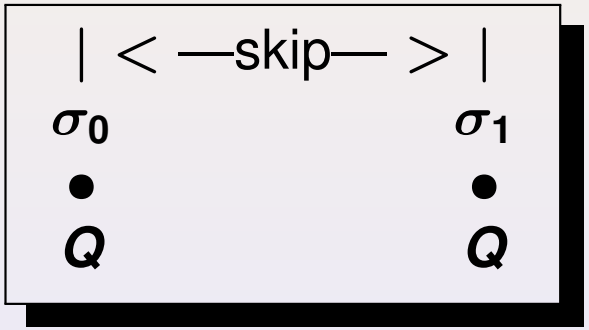
Informal Semantics (31)

Let σ_0, σ_1 and σ_2 be states and $\sigma_0(Q) = \text{tt}$, $\sigma_1(Q) = \text{tt}$ and $\sigma_3(Q) = \text{tt}$ then

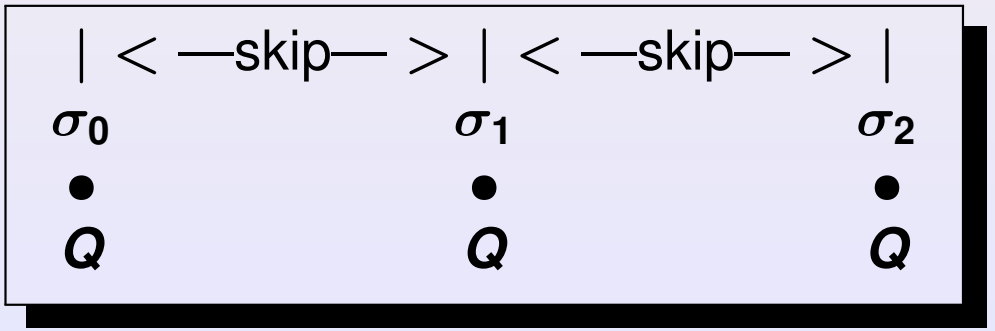
$[\![\Box\, Q]\!]_{\sigma_0} = \text{tt}$



$[\![\Box\, Q]\!]_{\sigma_0\sigma_1} = \text{tt}$



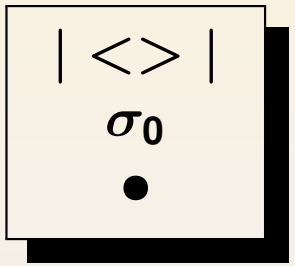
$[\![\Box\, Q]\!]_{\sigma_0\sigma_1\sigma_2} = \text{tt}$



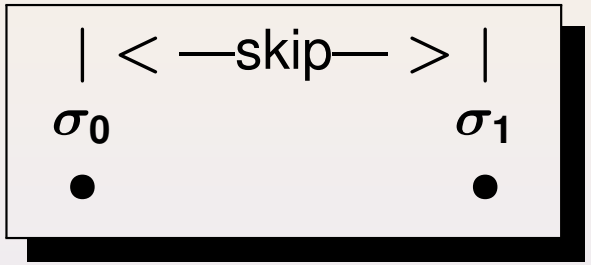
Informal Semantics (32)

Let σ_i be a state ($i \geq 0$) then

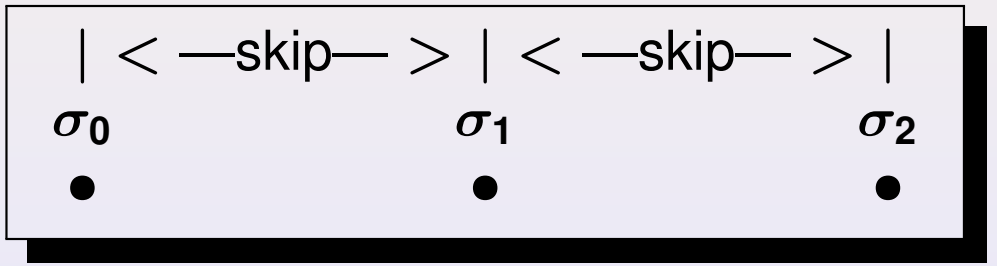
$[\![\text{skip}^*]\!]_{\sigma_0} = \text{tt}$



$[\![\text{skip}^*]\!]_{\sigma_0\sigma_1} = \text{tt}$

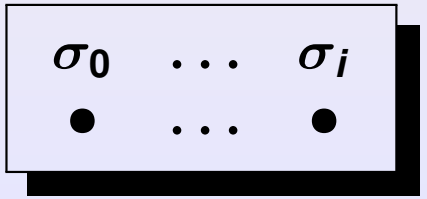


$[\![\text{skip}^*]\!]_{\sigma_0\sigma_1\sigma_2} = \text{tt}$



...

$[\![\text{skip}^*]\!]_{\sigma_0 \ldots \sigma_i} = \text{tt}$



Interval and Length

An interval σ is a (in)finite sequence of states

$$\sigma : \sigma_0\sigma_1\sigma_2\ldots$$

Let Σ^+ denote the set of all possible non-empty finite intervals.
Let Σ^ω denote the set of infinite intervals.

The length of an interval σ is denoted by $|\sigma|$ and is the number of states minus 1.

Example

$$\begin{array}{ll} \sigma = \sigma_0 & |\sigma| = 0 \\ \sigma = \sigma_0\sigma_1 & |\sigma| = 1 \\ \sigma = \sigma_0\sigma_1\ldots\sigma_n & |\sigma| = n \end{array}$$

Static and State Variables (34)

Static vs State Variables

- Static variables don't change their values within an interval.
- State variables can change their values within an interval.
  $\boldsymbol{Var} = \boldsymbol{StateVar} \bigcup \boldsymbol{StaticVar}$ and
  $\boldsymbol{StateVar} \bigcap \boldsymbol{StaticVar} = \emptyset$.

State variables are denoted by capital first symbols and static variables are denoted by small symbols.

Example

Let $\sigma : \sigma_0\sigma_1$ be an interval where

$$\begin{aligned} \sigma_0(Q) &= \text{tt} \\ \sigma_0(q) &= \text{ff} \\ \sigma_1(Q) &= \text{ff} \\ \sigma_1(q) &= \text{ff} \end{aligned}$$

Q is a state variable and q is a static variable.

Prefix, Suffix and Sub Interval (35)

Let $\sigma = \sigma_0\sigma_1\sigma_2\ldots$ be an interval then

- $\sigma_0\ldots\sigma_k$ (where $0 \leq k \leq |\sigma|$) denotes a prefix interval of σ
- $\sigma_k\ldots\sigma_{|\sigma|}$ (where $0 \leq k \leq |\sigma|$) denotes a suffix interval of σ
- $\sigma_k\ldots\sigma_l$ (where $0 \leq k \leq l \leq |\sigma|$) denotes a sub interval of σ

Example

Let $\sigma = \sigma_0\sigma_1\sigma_2\sigma_3$ be an interval then

$\sigma_0\sigma_1$	is a prefix interval of σ
$\sigma_1\sigma_2\sigma_3$	is a suffix interval of σ
$\sigma_1\sigma_2$	is a sub interval of σ

Semantics of PITL (36)

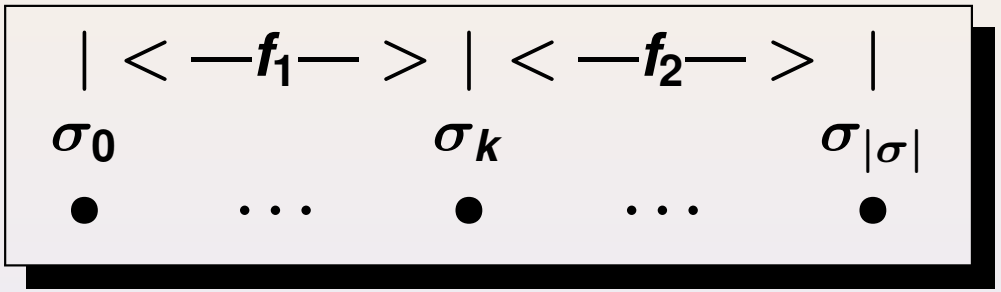
Let $[\![\ldots]\!]$ be the "meaning" function from $\mathit{PITL} \times (\Sigma^+ \cup \Sigma^\omega)$ to $\{\mathrm{tt}, \mathrm{ff}\}$ and let σ be an interval ($\sigma \in \Sigma^+ \cup \Sigma^\omega$) then

$$
\begin{array}{lcl}
[\![\mathrm{true}]\!]_\sigma & \widehat{=} & \mathrm{tt} \\
[\![q]\!]_\sigma & \widehat{=} & \sigma_0(q) \text{ and} \\
 & & \text{for all } 0 < i \leq |\sigma|, \sigma_i(q) = \sigma_0(q) \\
[\![Q]\!]_\sigma & \widehat{=} & \sigma_0(Q) \\
[\![\neg f]\!]_\sigma & \widehat{=} & \text{not } ([\![f]\!]_\sigma) \\
[\![f_1 \wedge f_2]\!]_\sigma & \widehat{=} & [\![f_1]\!]_\sigma \text{ and } [\![f_2]\!]_\sigma \\
[\![\mathrm{skip}]\!]_\sigma = \mathrm{tt} & \text{iff} & |\sigma| = 1
\end{array}
$$

Semantics of PITL (37)

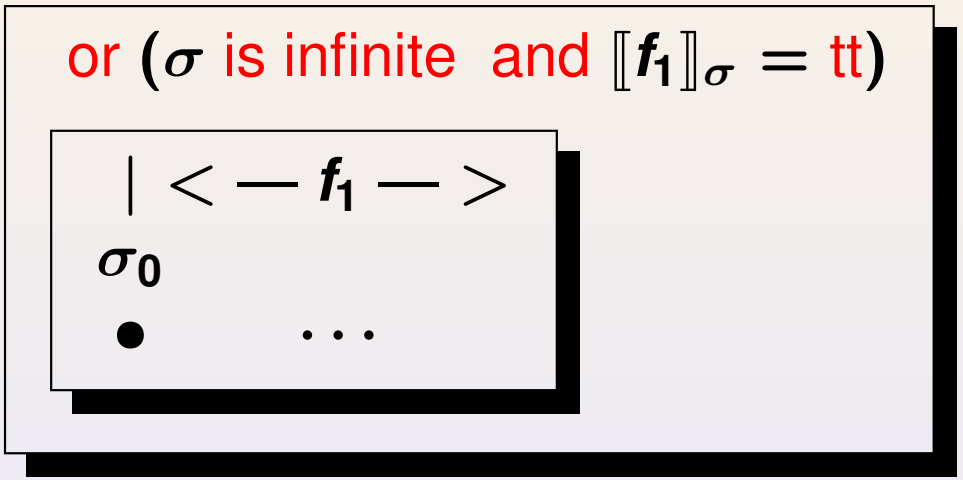
The semantics of 'chop' is as follows $[\![f_1 ; f_2]\!]_\sigma$ = tt iff

(exists k, s.t. $[\![f_1]\!]_{\sigma_0 \ldots \sigma_k}$ = tt and $[\![f_2]\!]_{\sigma_k \ldots \sigma_{|\sigma|}}$ = tt)



Interval σ is a fusion of two intervals $\sigma_0 \ldots \sigma_k$ (satisfies f_1) and $\sigma_k \ldots \sigma_{|\sigma|}$ (satisfies f_2). State σ_k is shared by both.

Semantics of PITL

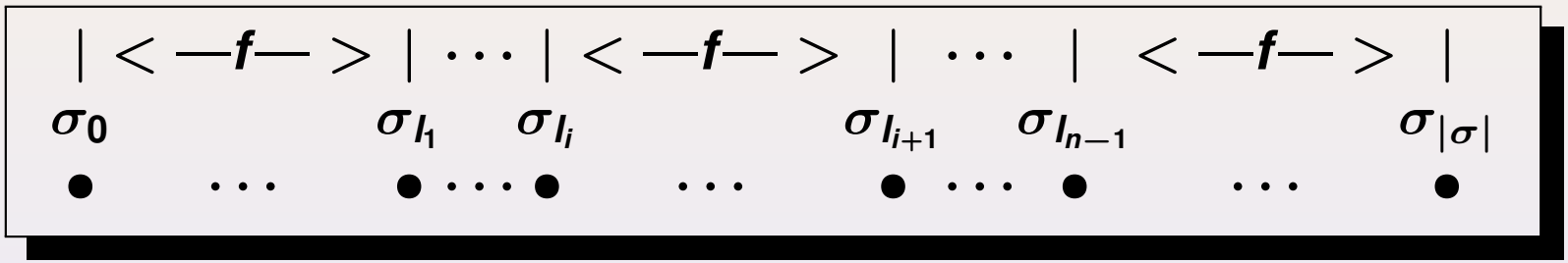


Interval σ is infinite and satisfies f_1, so f_2 is irrelevant.

Semantics of PITL

The semantics of 'chopstar' is as follows $[\![f^*]\!]_\sigma = \text{tt}$ iff
if σ is finite then

(exist $l_0, \ldots, l_n$ s.t. $l_0 = 0$ and $l_n = |\sigma|$ and
for all $0 \leq i < n, l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} = \text{tt}$)



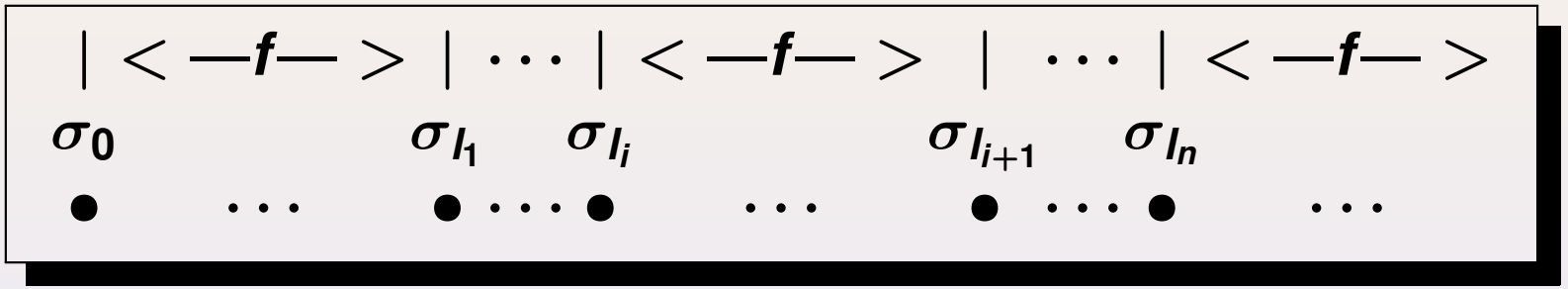
Finite interval σ is the fusion of a finite number of finite sub-intervals each satisfying f.

else

Semantics of PITL (40)

else

(exist $l_0, \ldots, l_n$ s.t. $l_0 = 0$ and
$[\![f]\!]_{\sigma_{l_n} \ldots |\sigma|} =$ tt and
for all $0 \leq i < n, l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} =$ tt)

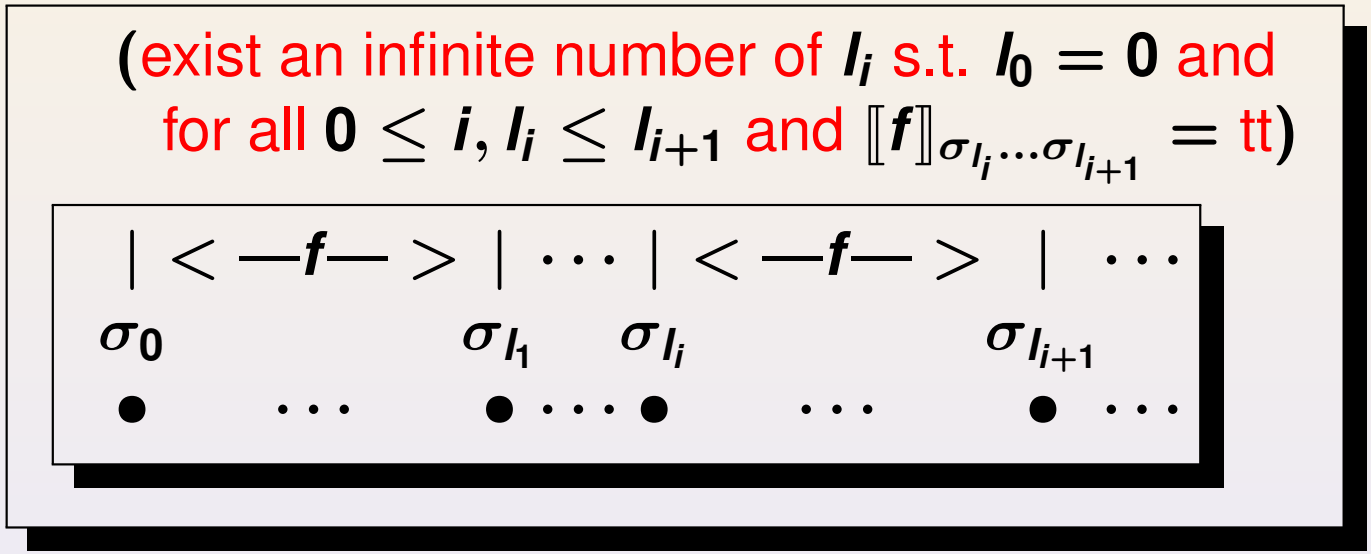


Infinite interval σ is the fusion of a finite number of sub-intervals each satisfying f. Each sub-interval is finite except the last one which is infinite.

or

Semantics of PITL (41)

or



Infinite interval σ is the fusion of an infinite number of finite sub-intervals each satisfying f.

Derived PITL formulae (42)

We will now introduce some derived temporal formulae.

$\bigcirc f$	$\widehat{=}$	skip ; f	next
$\textcircled{w} f$	$\widehat{=}$	$\neg \bigcirc \neg f$	weak next
more	$\widehat{=}$	$\bigcirc$true	non-empty interval
empty	$\widehat{=}$	$\neg$more	empty interval
inf	$\widehat{=}$	true ; false	infinite interval
finite	$\widehat{=}$	$\neg$inf	finite interval
$\Diamond f$	$\widehat{=}$	finite ; f	sometimes
$\Box f$	$\widehat{=}$	$\neg \Diamond \neg f$	always
$\textcircled{i}\!\!\Diamond f$	$\widehat{=}$	f ; true	some initial subinterval
$\boxed{\text{i}} f$	$\widehat{=}$	$\neg(\textcircled{i}\!\!\Diamond \neg f)$	all initial subintervals
$\textcircled{a}\!\!\Diamond f$	$\widehat{=}$	finite ; f ; true	some subinterval
$\boxed{\text{a}} f$	$\widehat{=}$	$\neg(\textcircled{a}\!\!\Diamond \neg f)$	all subintervals

Next and Weak Next (43)

- Next, f holds in the next state of the interval:

$$\bigcirc f \mathrel{\widehat{=}} \text{skip} ; f$$

$$[\![\bigcirc f]\!]_\sigma = \text{tt iff } |\sigma| > 0 \text{ and } ([\![f]\!]_{\sigma_1 \ldots \sigma_{|\sigma|}} = \text{tt})$$

- Weak Next, interval has only one state or f holds in the next state of the interval:

$$\circledcirc f \mathrel{\widehat{=}} \neg \bigcirc \neg f$$

$$[\![\circledcirc f]\!]_\sigma = \text{tt iff } |\sigma| = 0 \text{ or } ([\![f]\!]_{\sigma_1 \ldots \sigma_{|\sigma|}} = \text{tt})$$

More and Empty

- More, interval with at least two states:

$$\text{more} \mathrel{\widehat{=}} \bigcirc \text{true}$$

$$[\![\text{more}]\!]_\sigma = \text{tt iff } |\sigma| > \mathbf{0}$$

- Empty, interval with only one state:

$$\text{empty} \mathrel{\widehat{=}} \neg \text{more}$$

$$[\![\text{empty}]\!]_\sigma = \text{tt iff } |\sigma| = \mathbf{0}$$

Infinite and Finite

- Infinite, interval with an infinite number of states:

  $\mathit{inf} \mathrel{\widehat{=}} \mathit{true} \,;\, \mathit{false}$

  $[\![\mathit{inf}]\!]_\sigma = \mathit{tt}$ iff σ is infinite

- Finite, interval with a finite number of states:

  $\mathit{finite} \mathrel{\widehat{=}} \neg \mathit{inf}$

  $[\![\mathit{finite}]\!]_\sigma = \mathit{tt}$ iff σ is finite

Sometimes and Always (46)

- Sometimes, there exist a suffix interval that satisfies f:

$$\Diamond f \;\widehat{=}\; \text{finite} \,;\, f$$

$$[\![\Diamond f]\!]_\sigma = \text{tt iff } (\text{exists a } 0 \leq k \leq |\sigma|, \text{ s.t. } [\![f]\!]_{\sigma_k \ldots \sigma_{|\sigma|}} = \text{tt})$$

- Always, all suffix intervals satisfy f:

$$\Box f \;\widehat{=}\; \neg \Diamond \neg f$$

$$[\![\Box f]\!]_\sigma = \text{tt iff } (\text{for all } 0 \leq k \leq |\sigma|, \text{ s.t. } [\![f]\!]_{\sigma_k \ldots \sigma_{|\sigma|}} = \text{tt})$$

Diamond-i and Box-i (47)

- Diamond-i, there exists a prefix interval that satisfies f:

$$\diamondsuit_i f \;\widehat{=}\; f \,;\, \text{true}$$

$$[\![\diamondsuit_i f]\!]_\sigma = \text{tt iff } (\text{exists a } 0 \leq k \leq |\sigma|, \text{ s.t. } [\![f]\!]_{\sigma_0 \ldots \sigma_k} = \text{tt})$$

- Box-i, all prefix intervals satisfy f:

$$\boxdot_i f \;\widehat{=}\; \neg \diamondsuit_i \neg f$$

$$[\![\boxdot_i f]\!] = \text{tt iff } (\text{for all } 0 \leq k \leq |\sigma|, \text{ s.t. } [\![f]\!]_{\sigma_0 \ldots \sigma_k} = \text{tt})$$

Diamond-a and Box-a (48)

- Diamond-a, there exists a sub-interval that satisfies ***f***:

$$\Diamond_a \, f \;\widehat{=}\; \text{finite} \,;\, f \,;\, \text{true}$$

$$[\![\Diamond_a \, f]\!] = \text{tt iff } (\text{exist } 0 \leq k \leq l \leq |\sigma| \text{ s.t. } [\![f]\!]_{\sigma_k \ldots \sigma_l} = \text{tt})$$

- Box-a, all sub-intervals satisfy ***f***:

$$\Box_a \, f \;\widehat{=}\; \neg \Diamond_a \neg f$$

$$[\![\Box_a \, f]\!] = \text{tt iff } (\text{for all } 0 \leq k \leq l \leq |\sigma| \text{ s.t. } [\![f]\!]_{\sigma_k \ldots \sigma_l} = \text{tt})$$

Programming constructs (49)

- Following are derived temporal formulae that are "programming constructs":

if f_0 then f_1 else f_2	$\widehat{=}$	$(f_0 \wedge f_1) \vee (\neg f_0 \wedge f_2)$	if then else
if f_0 then f_1	$\widehat{=}$	if f_0 then f_1 else empty	if then
halt f	$\widehat{=}$	$\Box(\text{empty} \equiv f)$	halt when
keep f	$\widehat{=}$	$\boxed{a}(\text{skip} \supset f)$	all unit sub-intervals
while f_0 do f_1	$\widehat{=}$	$(f_0 \wedge f_1)^* \wedge \text{fin} \neg f_0$	while loop
repeat f_0 until f_1	$\widehat{=}$	$f_0 ; (\text{while } \neg f_1 \text{ do } f_0)$	repeat loop

Exercises

Exercise

Let $\sigma = \sigma_0\sigma_1\sigma_2\sigma_3$
Give all prefix intervals of σ
Give all suffix intervals of σ
Give all sub-intervals of σ

Exercise

Exercise

Give the formal semantics of following formulae:

- $\Box(\text{empty} \supset \boldsymbol{f})$
- $\boxed{\text{i}}(\text{empty} \supset \boldsymbol{f})$
- $\boxed{\text{a}}(\text{empty} \supset \boldsymbol{f})$
- $\Box(\text{more} \supset \boldsymbol{f})$
- $\boxed{\text{i}}(\text{more} \supset \boldsymbol{f})$
- $\boxed{\text{a}}(\text{more} \supset \boldsymbol{f})$
- $\Box(\text{skip} \supset \boldsymbol{f})$
- $\boxed{\text{i}}(\text{skip} \supset \boldsymbol{f})$
- $\boxed{\text{a}}(\text{skip} \supset \boldsymbol{f})$

Exercise

Exercise

Give the informal semantics (picture) of following formulae:

if f_0 then f_1 else f_2
fin f
halt f
keep f
while f_0 do f_1
repeat f_0 until f_1

Part III

First Order Logic

Outline

First Order Logic (55)

Syntax of Integer Expressions:

- Integer values: $\mathbf{0}, \mathbf{1}, \ldots,$
- Static integer variable: $\boldsymbol{a}, \boldsymbol{b}, \ldots$
- State integer variable: $\boldsymbol{A}, \boldsymbol{B}, \ldots$
- Integer operators: $+, -, *, **, \mathit{mod}, \mathit{div}, \ldots$

Syntax of Formulae:

- Boolean values: true, false
- Boolean predicates: $\boldsymbol{e_1} = \boldsymbol{e_2}, \boldsymbol{e_1} \neq \boldsymbol{e_2}, \boldsymbol{e_1} < \boldsymbol{e_2}, \boldsymbol{e_1} \leq \boldsymbol{e_2}, \boldsymbol{e_1} > \boldsymbol{e_2}, \boldsymbol{e_1} \geq \boldsymbol{e_2}, \ldots$
- Boolean operators: $\wedge, \vee, \neg, \supset, \equiv,$
  $\forall \boldsymbol{v} \bullet \boldsymbol{f}$ (for all $\boldsymbol{v}$ such that $\boldsymbol{f}$ holds)

First Order Logic (56)

Syntax of Integer Expressions in BNF:

$$e ::= z \mid a \mid A \mid g(e_1, \ldots, e_n)$$

where z is an integer constant and g an integer operator.

Syntax of First Order Formulae in BNF:

$$f ::= \text{true} \mid q \mid Q \mid h(e_1, \ldots, e_n) \mid \neg f \mid f_1 \wedge f_2 \mid \forall v \bullet f$$

where h is a Boolean predicate.

Derived formulae:

$$\exists v \bullet f \;\widehat{=}\; \neg \forall v \bullet \neg f$$

First Order Logic Examples (57)

Example

$$4 > 3$$
$$A + 5 \leq B \wedge B = D - 7$$
$$P \equiv (A + 5 \leq B)$$
$$Q \equiv (B = D - 7)$$
$$\forall A \bullet (A + 1 > A)$$
$$A = 8 \wedge \exists b \bullet (A = 2 * *b)$$

State

A ***State*** is a union of

- an integer state $\boldsymbol{State^e}$ which is a mapping from the set of integer variables $\boldsymbol{Var^e}$ to the set of integer values ***Val*** and
- a Boolean state $\boldsymbol{State^b}$ which is a mapping from the set of propositional variable $\boldsymbol{Var^b}$ to the set of Boolean values ***Bool***.

$$\boldsymbol{State} : (\boldsymbol{Var^e} \rightarrow \boldsymbol{Val}) \cup (\boldsymbol{Var^b} \rightarrow \boldsymbol{Bool})$$

where $\boldsymbol{Var^e} \cap \boldsymbol{Var^b} = \emptyset$.
We will use $\sigma_0, \sigma_1, \sigma_2, \ldots$ to denote states and $\boldsymbol{\Sigma}$ to denote the set of all possible states.

Example

Let $\boldsymbol{p}$ be a Boolean variable and $\boldsymbol{A}$ be an integer variable then σ_0 s.t. $\sigma_0(\boldsymbol{p}) = \mathrm{tt}$ and $\sigma_0(\boldsymbol{A}) = \mathbf{5}$ is a state.

Semantics of expressions (59)

Let $[\![\ldots]\!]^{e}$ be the "meaning" (semantic) function from ***Expressions*** $\times$ Σ to ***Val*** (integer values) and let σ_0 be a state then

$$
\begin{array}{lcl}
[\![z]\!]^{e}_{\sigma_0} & = & z \\
[\![a]\!]^{e}_{\sigma_0} & = & \sigma_0(a) \\
[\![A]\!]^{e}_{\sigma_0} & = & \sigma_0(A) \\
[\![g(e_1,\ldots,e_n)]\!]^{e}_{\sigma_0} & = & g([\![e_1]\!]^{e}_{\sigma_0},\ldots,[\![e_n]\!]^{e}_{\sigma_0})
\end{array}
$$

Example

$$
\begin{array}{lcl}
[\![Account]\!]^{e}_{\sigma_0} & = & \sigma_0(Account) \\
[\![deposit]\!]^{e}_{\sigma_0} & = & \sigma_0(deposit) \\
[\![Account + deposit]\!]^{e}_{\sigma_0} & = & [\![Account]\!]^{e}_{\sigma_0} + [\![deposit]\!]^{e}_{\sigma_0} \\
 & = & \sigma_0(Account) + \sigma_0(deposit)
\end{array}
$$

Semantics of formulae (60)

Let $[\![\ldots]\!]$ be the "meaning" function from $\mathbf{Formulae} \times \mathbf{\Sigma}$ to **Bool** (set of Boolean values, $\{\mathrm{tt}, \mathrm{ff}\}$) and let σ_0 be a state then

$$
\begin{array}{lll}
[\![\mathrm{true}]\!]_{\sigma_0} & = & \mathrm{tt} \\
[\![q]\!]_{\sigma_0} & = & \sigma_0(q) \\
[\![Q]\!]_{\sigma_0} & = & \sigma_0(Q) \\
[\![h(e_1,\ldots,e_n)]\!]_{\sigma_0} = \mathrm{tt} \text{ iff} & & h([\![e_1]\!]^e_{\sigma_0},\ldots,[\![e_n]\!]^e_{\sigma_0}) \\
[\![\neg f]\!]_{\sigma_0} = \mathrm{tt} & \text{iff} & \text{not } ([\![f]\!]_{\sigma_0} = \mathrm{tt}) \\
[\![f_1 \wedge f_2]\!]_{\sigma_0} = \mathrm{tt} & \text{iff} & ([\![f_1]\!]_{\sigma_0} = \mathrm{tt}) \text{ and } ([\![f_2]\!]_{\sigma_0} = \mathrm{tt})
\end{array}
$$

Example

$$
\begin{array}{ll}
([\![\neg(Account < 0)]\!]_{\sigma_0} = \mathrm{tt}) & \text{iff} \\
\text{not } ([\![Account < 0]\!]_{\sigma_0} = \mathrm{tt}) & \text{iff} \\
\text{not } ([\![Account]\!]^e_{\sigma_0} < [\![0]\!]^e_{\sigma_0}) & \text{iff} \\
\text{not } (\sigma_0(Account) < 0) &
\end{array}
$$

Example

Example

$$(\llbracket \mathit{Account} = 50 \wedge \mathit{Deposit} \geq 0 \rrbracket_{\sigma_0} = \mathrm{tt}) \quad \text{iff}$$
$$(\llbracket \mathit{Account} = 50 \rrbracket_{\sigma_0} = \mathrm{tt}) \text{ and } (\llbracket \mathit{Deposit} \geq 0 \rrbracket_{\sigma_0} = \mathrm{tt}) \quad \text{iff}$$
$$(\llbracket \mathit{Account} \rrbracket^{e}_{\sigma_0} = \llbracket 50 \rrbracket^{e}_{\sigma_0}) \text{ and } (\llbracket \mathit{Deposit} \rrbracket^{e}_{\sigma_0} \geq \llbracket 0 \rrbracket^{e}_{\sigma_0}) \quad \text{iff}$$
$$\sigma_0(\mathit{Account}) = 50 \text{ and } \sigma_0(\mathit{Deposit}) \geq 0$$

Semantics of formulae

Let $\sigma_0 \sim_{\boldsymbol{v}} \sigma_0'$ denote that the states σ_0 and σ_0' are identical with the possible exception of the mapping for the variable $\boldsymbol{v}$.

Example

Let σ_0 and σ_0' be states.

- Let $\sigma_0(\boldsymbol{Cash}) = \mathbf{0}$ and $\sigma_0(\boldsymbol{Deposit}) = \mathbf{2}$
- Let $\sigma_0'(\boldsymbol{Cash}) = \mathbf{0}$ and $\sigma_0'(\boldsymbol{Deposit}) = \mathbf{3}$

Then $\sigma_0 \sim_{\boldsymbol{Deposit}} \sigma_0'$.

Semantics of formulae

The semantics of $\forall \boldsymbol{v} \bullet \boldsymbol{f}$ is defined in terms of $\sigma_0 \sim_{\boldsymbol{v}} \sigma_0'$

$$[\![\forall \boldsymbol{v} \bullet \boldsymbol{f}]\!] = \text{tt} \quad \text{iff} \quad (\text{for all } \sigma_0' \text{ s.t. } \sigma_0 \sim_{\boldsymbol{v}} \sigma_0', [\![\boldsymbol{f}]\!]_{\sigma_0'} = \text{tt})$$

Example

$$[\![\forall \boldsymbol{Deposit} \bullet \boldsymbol{Deposit} \geq \mathbf{0}]\!]_{\sigma_0} = \text{tt} \quad \text{iff}$$
$$(\text{for all } \sigma_0' \text{ s.t. } \sigma_0 \sim_{\boldsymbol{Deposit}} \sigma_0', [\![\boldsymbol{Deposit} \geq \mathbf{0}]\!]_{\sigma_0} = \text{tt}) \quad \text{iff}$$
$$(\text{for all } \sigma_0' \text{ s.t. } \sigma_0 \sim_{\boldsymbol{Deposit}} \sigma_0', \sigma_0'(\boldsymbol{Deposit}) \geq \mathbf{0})$$

Exercises

Exercise

Give the semantics of the following formulae

$$4 > 3$$
$$A + 5 \leq B \wedge B = D - 7$$
$$P \equiv (A + 5 \leq B)$$
$$Q \equiv (B = D - 7)$$
$$\forall A \bullet (A + 1 > A)$$
$$A = 8 \wedge \exists b \bullet (A = 2 * *b)$$

Part IV

First Order ITL

Outline

First Order ITL (67)

Syntax of Integer Expressions:

- Integer values: $\mathbf{0}, \mathbf{1}, \ldots,$
- Static integer variable: $\boldsymbol{a}, \boldsymbol{b}, \ldots$
- State integer variable: $\boldsymbol{A}, \boldsymbol{B}, \ldots$
- Integer operators: $+, -, *, **, \mathit{mod}, \mathit{div}, \ldots$
- Temporal integer variable: $\bigcirc \boldsymbol{A}$ (next A), $\bigcirc \boldsymbol{B}, \ldots,$ fin $\boldsymbol{A}$ (fin A), fin $\boldsymbol{B}, \ldots$

Syntax of Formulae:

- Boolean values: true, false
- Boolean static variables: $\boldsymbol{p}, \boldsymbol{q}, \ldots$
- Boolean state variables: $\boldsymbol{P}, \boldsymbol{Q}, \ldots$
- Boolean predicates: $\boldsymbol{e_1} = \boldsymbol{e_2}, \boldsymbol{e_1} \neq \boldsymbol{e_2}, \boldsymbol{e_1} < \boldsymbol{e_2}, \boldsymbol{e_1} \leq \boldsymbol{e_2}, \boldsymbol{e_1} > \boldsymbol{e_2}, \boldsymbol{e_1} \geq \boldsymbol{e_2}, \ldots$
- Boolean operators: $\wedge, \vee, \neg, \supset, \equiv, \forall \boldsymbol{v} \bullet \boldsymbol{f}$
- Temporal operators: skip, ;,* , $\bigcirc, \Box, \Diamond, \ldots$

First Order ITL (68)

Syntax of Integer Expressions in BNF:

$$e ::= z \mid a \mid A \mid g(e_1, \ldots, e_n) \mid \bigcirc A \mid \text{fin } A$$

where z is an integer constant and g an integer operator.

Syntax of First Order Formulae in BNF:

$$f ::= \text{true} \mid q \mid Q \mid h(e_1, \ldots, e_n) \mid \neg f \mid f_1 \wedge f_2 \mid \forall v \bullet f \mid \text{skip} \mid f_1 ; f_2 \mid f^*$$

where h is a Boolean predicate.

First Order ITL

Derived formulae:

$$
\begin{array}{lcl}
\text{false} & \widehat{=} & \neg\text{true} \\
f_1 \vee f_2 & \widehat{=} & \neg(\neg f_1 \wedge \neg f_2) \\
f_1 \supset f_2 & \widehat{=} & \neg f_1 \vee f_2 \\
f_1 \equiv f_2 & \widehat{=} & (f_1 \supset f_2) \wedge (f_2 \supset f_1) \\
\exists v \bullet f & \widehat{=} & \neg\forall v \bullet \neg f \\
\bigcirc f & \widehat{=} & \text{skip} ; f \\
\text{inf} & \widehat{=} & \text{true} ; \text{false} \\
\text{finite} & \widehat{=} & \neg\text{inf} \\
\Diamond f & \widehat{=} & \text{finite} ; f \\
\Box f & \widehat{=} & \neg(\Diamond \neg f) \\
\ldots & &
\end{array}
$$

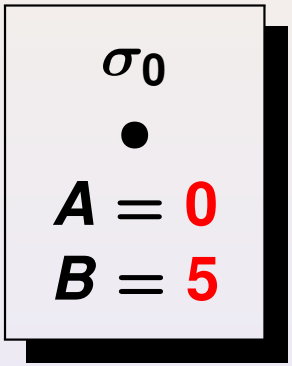
First Order ITL Examples (70)

Example

$$(\bigcirc \boldsymbol{B}) = \mathbf{0} \wedge (\text{skip} \,;\, \boldsymbol{A} = \mathbf{1})$$
$$\boldsymbol{A} = \mathbf{0} \wedge ((\boldsymbol{B} = \mathbf{1} \wedge \text{skip}) \,;\, (\boldsymbol{A} = \mathbf{1} \wedge \boldsymbol{B} = \mathbf{0}))$$
$$\boldsymbol{A} = \mathbf{0} \wedge \bigcirc(\boldsymbol{A} = \mathbf{1}) \wedge \bigcirc\bigcirc(\boldsymbol{A} = \mathbf{2})$$
$$(\Diamond \boldsymbol{A} = \mathbf{0}) \wedge (\Diamond \boldsymbol{B} = \mathbf{0})$$
$$\Box \boldsymbol{A} = \mathbf{5}$$
$$(\boldsymbol{A} = \mathbf{0} \wedge \text{skip})^* \wedge (\text{fin } \boldsymbol{A}) = \mathbf{0}$$

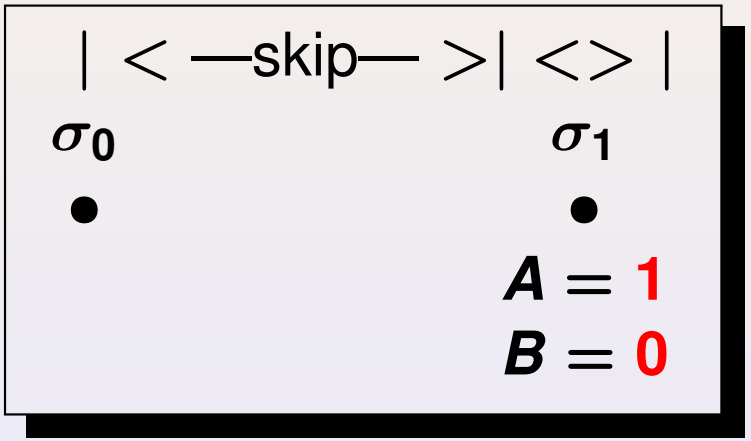
Informal Semantics (71)

In first order logic semantics is given wrt a state:
Let σ_0 be a state and $\sigma_0(A) = 0$ and $\sigma_0(B) = 5$ then
$[\![A = 0 \vee B = 1]\!]_{\sigma_0} = \text{tt}$



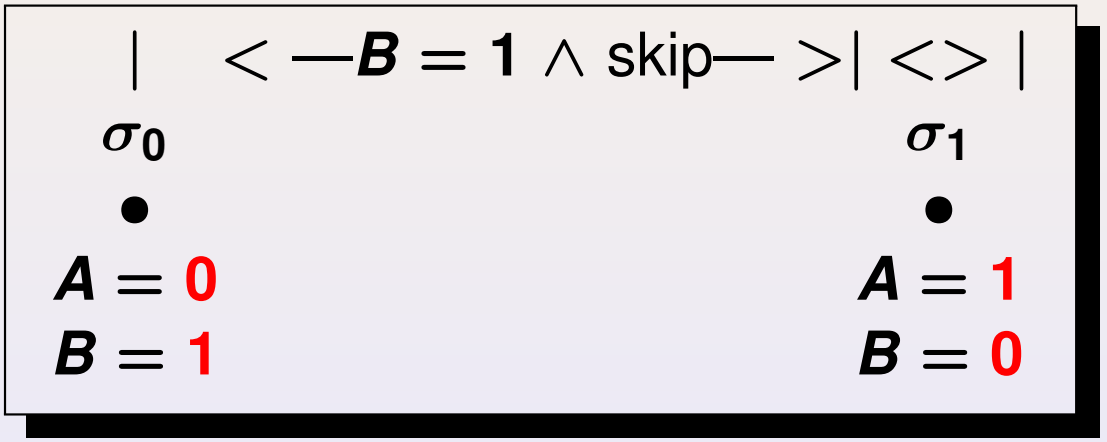
Informal Semantics (72)

In first order ITL semantics is given wrt a sequence of states:
Let σ_0 and σ_1 be states and $\sigma_1(B) = 0$ and $\sigma_1(A) = 1$ then
$[\![(\bigcirc B) = 0 \wedge (\text{skip} ; A = 1)]\!]_{\sigma_0\sigma_1} = \text{tt}$



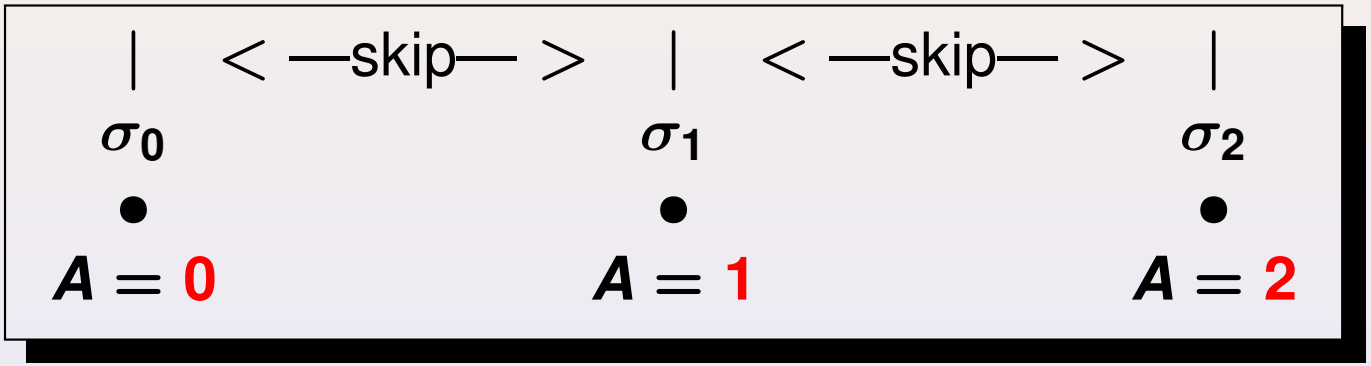
Informal Semantics (73)

Let σ_0 and σ_1 be states and $\sigma_0(A) = 0$, $\sigma_0(B) = 1$, $\sigma_1(A) = 1$ and $\sigma_1(B) = 0$ then
$[\![A = 0 \wedge ((B = 1 \wedge \text{skip}) ; (A = 1 \wedge B = 0))]\!]_{\sigma_0\sigma_1} = \text{tt}$



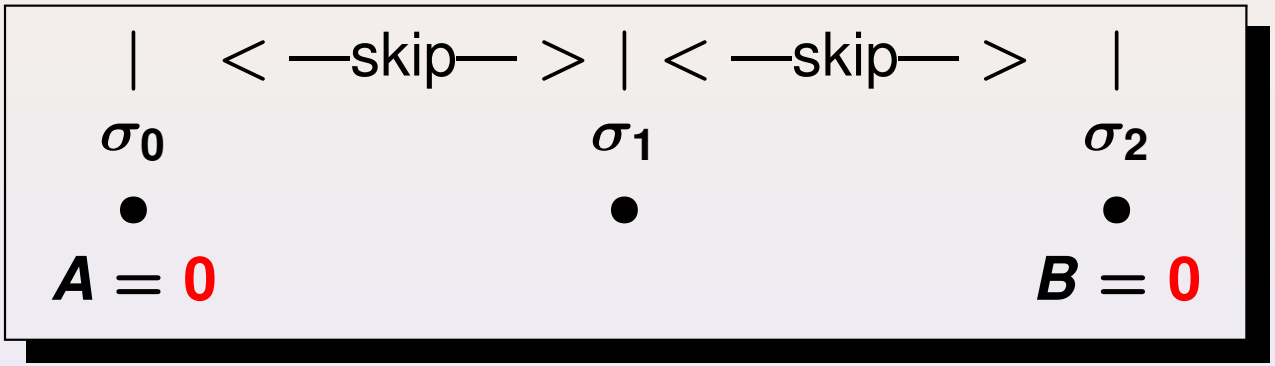
Informal Semantics (74)

Let σ_0, σ_1 and σ_2 be states and $\sigma_0(A) = 0$, $\sigma_1(A) = 1$ and $\sigma_2(A) = 2$ then
$[\![A = 0 \wedge \bigcirc(A = 1) \wedge \bigcirc\bigcirc(A = 2)]\!]_{\sigma_0\sigma_1\sigma_2} = \mathrm{tt}$



Informal Semantics (75)

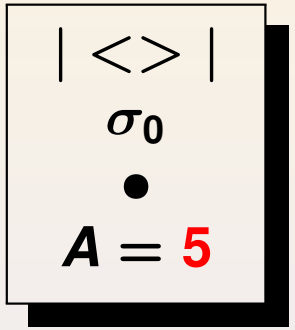
Let σ_0, σ_1 and σ_2 be states and $\sigma_0(A) = 0$ and $\sigma_2(B) = 0$ then $[\![(\Diamond A = 0) \wedge (\Diamond B = 0)]\!]_{\sigma_0\sigma_1\sigma_2} = \text{tt}$



Informal Semantics (76)

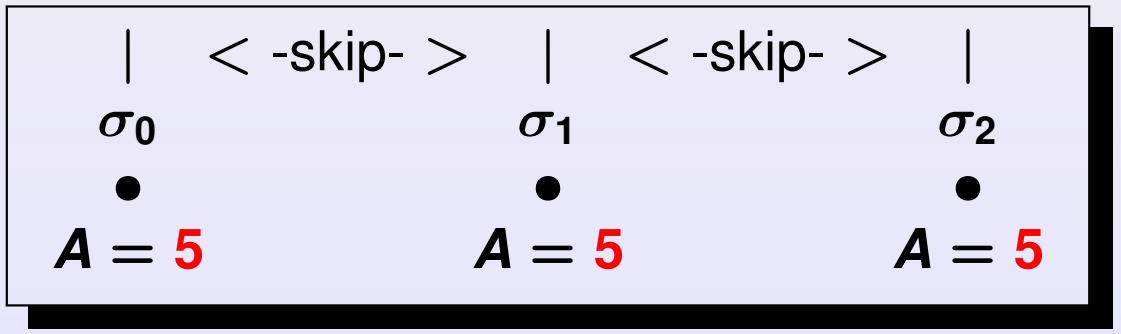
Let σ_0, σ_1 and σ_2 be states and $\sigma_0(A) = 5$, $\sigma_1(A) = 5$ and $\sigma_3(A) = 5$ then

$[\![\Box\, A = 5]\!]_{\sigma_0} = \text{tt}$



$[\![\Box\, A = 5]\!]_{\sigma_0\sigma_1} = \text{tt}$

$[\![\Box\, A = 5]\!]_{\sigma_0\sigma_1\sigma_2} = \text{tt}$



Informal Semantics

Let σ_i be a state ($i \geq i$) and $\sigma_i(A) = 0$ then

$$
\begin{array}{lcl}
[\![(A = 0 \wedge \text{skip})^* \wedge (\text{fin } A) = 0]\!]_{\sigma_0} & = & \text{tt} \\
[\![(A = 0 \wedge \text{skip})^* \wedge (\text{fin } A) = 0]\!]_{\sigma_0\sigma_1} & = & \text{tt} \\
[\![(A = 0 \wedge \text{skip})^* \wedge (\text{fin } A) = 0]\!]_{\sigma_0\sigma_1\sigma_2} & = & \text{tt} \\
\ldots & & \\
[\![(A = 0 \wedge \text{skip})^* \wedge (\text{fin } A) = 0]\!]_{\sigma_0 \ldots \sigma_i} & = & \text{tt}
\end{array}
$$

Note: looks similar to $\Box\, A = 0$ but is it really equal?

Semantics of expressions (78)

Let $[\![\ldots]\!]^{e}$ be the "meaning" (semantic) function from ***Expressions*** $\times$ $(\Sigma^{+} \cup \Sigma^{\omega})$ to ***Val*** and let $\sigma = \sigma_0\sigma_1\ldots$ be an interval then

$$
\begin{array}{lcl}
[\![z]\!]^{e}_{\sigma} & = & z \\
[\![a]\!]^{e}_{\sigma} & = & \sigma_0(a) \text{ and} \\
 & & \text{for all } 0 < i \leq |\sigma|, \sigma_i(a) = \sigma_0(a) \\
[\![A]\!]^{e}_{\sigma} & = & \sigma_0(A) \\
[\![g(e_1,\ldots,e_n)]\!]^{e}_{\sigma} & = & g([\![e_1]\!]^{e}_{\sigma},\ldots,[\![e_n]\!]^{e}_{\sigma}) \\
[\![\bigcirc A]\!]^{e}_{\sigma} & = & \begin{cases} \sigma_1(A) & \text{implies } |\sigma| > 0 \\ \text{choose-any-from}(Val) & \text{otherwise} \end{cases} \\
[\![\text{fin } A]\!]^{e}_{\sigma} & = & \begin{cases} \sigma_{|\sigma|}(A) & \text{implies } \sigma \text{ is finit} \\ \text{choose-any-from}(Val) & \text{otherwise} \end{cases}
\end{array}
$$

Example of semantics

Example

$$[\![\mathit{Account}]\!]^{e}_{\sigma} = \sigma_0(\mathit{Account})$$

$[\![\mathit{deposit}]\!]^{e}_{\sigma} = \sigma_0(\mathit{deposit})$ and
for all i s.t. $0 < i \leq |\sigma|, \sigma_i(\mathit{deposit}) = \sigma_0(\mathit{deposit})$

$$[\![\mathit{Account} + \mathit{deposit}]\!]^{e}_{\sigma} = [\![\mathit{Account}]\!]^{e}_{\sigma} + [\![\mathit{deposit}]\!]^{e}_{\sigma} = \sigma_0(\mathit{Account}) + \sigma_0(\mathit{deposit})$$

and for all i s.t. $0 < i \leq |\sigma|, \sigma_i(\mathit{deposit}) = \sigma_0(\mathit{deposit})$

Semantics of formulae

Let $[\![\ldots]\!]$ be the "meaning" function from ***Formulae*** $\times$ $(\Sigma^+ \cup \Sigma^\omega)$ to ***Bool*** (set of Boolean values, $\{\mathrm{tt}, \mathrm{ff}\}$) and let $\sigma = \sigma_0\sigma_1\ldots$ be an interval then

$[\![\mathrm{true}]\!]_\sigma$	$=$	tt
$[\![q]\!]_\sigma$	$=$	$\sigma_0(q)$ and for all $0 < i \leq \lvert\sigma\rvert$, $\sigma_i(q) = \sigma_0(q)$
$[\![Q]\!]_\sigma$	$=$	$\sigma_0(Q)$
$[\![h(e_1,\ldots,e_n)]\!]_\sigma = \mathrm{tt}$	iff	$h([\![e_1]\!]^e_\sigma,\ldots,[\![e_n]\!]^e_\sigma)$
$[\![\neg f]\!]_\sigma = \mathrm{tt}$	iff	not $([\![f]\!]_\sigma = \mathrm{tt})$
$[\![f_1 \wedge f_2]\!]_\sigma = \mathrm{tt}$	iff	$([\![f_1]\!]_\sigma = \mathrm{tt})$ and $([\![f_2]\!]_\sigma = \mathrm{tt})$
$[\![\mathrm{skip}]\!]_\sigma = \mathrm{tt}$	iff	$\lvert\sigma\rvert = 1$

Example

Example

$$
\begin{array}{ll}
([\![\neg(\mathit{Account} < 0)]\!]_\sigma = \mathrm{tt}) & \text{iff} \\
\text{not } ([\![\mathit{Account} < 0]\!]_\sigma = \mathrm{tt}) & \text{iff} \\
\text{not } ([\![\mathit{Account}]\!]^e_\sigma < [\![0]\!]^e_\sigma) & \text{iff} \\
\text{not } (\sigma_0(\mathit{Account}) < 0)
\end{array}
$$

Example

$$
\begin{array}{ll}
([\![\mathit{Account} = 50 \wedge \mathit{Deposit} \geq 0]\!]_\sigma = \mathrm{tt}) & \text{iff} \\
([\![\mathit{Account} = 50]\!]_\sigma = \mathrm{tt}) \text{ and } ([\![\mathit{Deposit} \geq 0]\!]_\sigma = \mathrm{tt}) & \text{iff} \\
([\![\mathit{Account}]\!]^e_\sigma = [\![50]\!]^e_\sigma) \text{ and } ([\![\mathit{Deposit}]\!]^e_\sigma \geq [\![0]\!]^e_\sigma) & \text{iff} \\
\sigma_0(\mathit{Account}) = 50 \text{ and } \sigma_0(\mathit{Deposit}) \geq 0
\end{array}
$$

Semantics of formulae

Let $\sigma \sim_{v} \sigma'$ denote that the intervals σ and σ' are identical with the possible exception of the mapping for the variable v.

Example

Let σ and σ' be intervals.

- Let $\sigma_0(\mathit{Cash}) = 0$ and $\sigma_0(\mathit{Deposit}) = 2$.
  Let $\sigma_1(\mathit{Cash}) = 1$ and $\sigma_1(\mathit{Deposit}) = 4$.
  Let $\sigma_2(\mathit{Cash}) = 1$ and $\sigma_2(\mathit{Deposit}) = 3$.
- Let $\sigma'_0(\mathit{Cash}) = 0$ and $\sigma'_0(\mathit{Deposit}) = 3$.
  Let $\sigma'_1(\mathit{Cash}) = 1$ and $\sigma'_0(\mathit{Deposit}) = 2$.
  Let $\sigma'_2(\mathit{Cash}) = 1$ and $\sigma'_2(\mathit{Deposit}) = 3$.

Then $\sigma \sim_{\mathit{Deposit}} \sigma'$.

Semantics of formulae

The semantics of $\forall \boldsymbol{v} \bullet \boldsymbol{f}$ is defined in terms of $\sigma \sim_{\boldsymbol{v}} \sigma'$

$$[\![\forall \boldsymbol{v} \bullet \boldsymbol{f}]\!] = \text{tt} \quad \text{iff} \quad (\text{for all } \sigma' \text{ s.t. } \sigma \sim_{\boldsymbol{v}} \sigma', [\![\boldsymbol{f}]\!]_{\sigma'} = \text{tt})$$

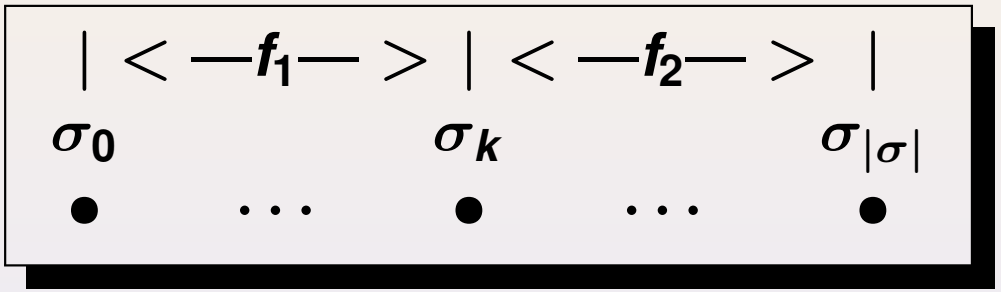
Example

$$[\![\forall \boldsymbol{Deposit} \bullet \boldsymbol{Deposit} \geq \mathbf{0}]\!]_{\sigma} = \text{tt} \quad \text{iff}$$
$$(\text{for all } \sigma' \text{ s.t. } \sigma \sim_{\boldsymbol{Deposit}} \sigma', [\![\boldsymbol{Deposit} \geq \mathbf{0}]\!]_{\sigma} = \text{tt}) \quad \text{iff}$$
$$(\text{for all } \sigma' \text{ s.t. } \sigma \sim_{\boldsymbol{Deposit}} \sigma', \sigma'_{\mathbf{0}}(\boldsymbol{Deposit}) \geq \mathbf{0})$$

Semantics of formulae (84)

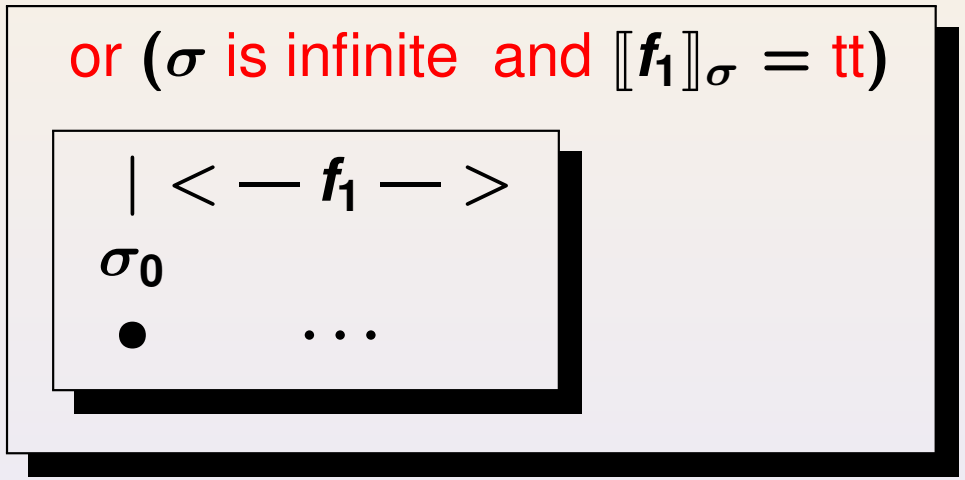
The semantics of 'chop' is as follows $[\![f_1 ; f_2]\!]_\sigma =$ tt iff

(exists k, s.t. $[\![f_1]\!]_{\sigma_0 \ldots \sigma_k} =$ tt and $[\![f_2]\!]_{\sigma_k \ldots \sigma_{|\sigma|}} =$ tt)



Interval σ is a fusion of two intervals $\sigma_0 \ldots \sigma_k$ (satisfies f_1) and $\sigma_k \ldots \sigma_{|\sigma|}$ (satisfies f_2). State σ_k is shared by both.

Semantics of formulae



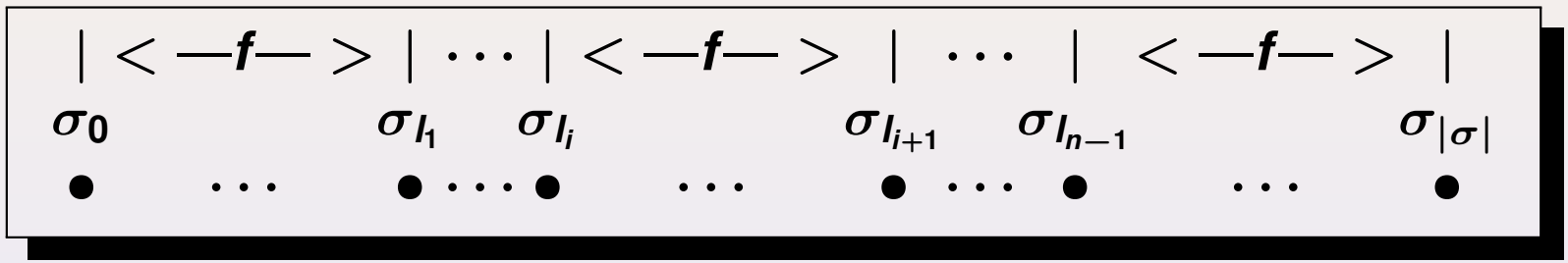
Interval σ is infinite and satisfies f_1, so f_2 is irrelevant.

Semantics of formulae (86)

The semantics of 'chopstar' is as follows $[\![f^*]\!] = \text{tt}$ iff

if σ is finite then

(exist $l_0, \ldots, l_n$ s.t. $l_0 = 0$ and $l_n = |\sigma|$ and
for all $0 \leq i < n, l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} = \text{tt}$)



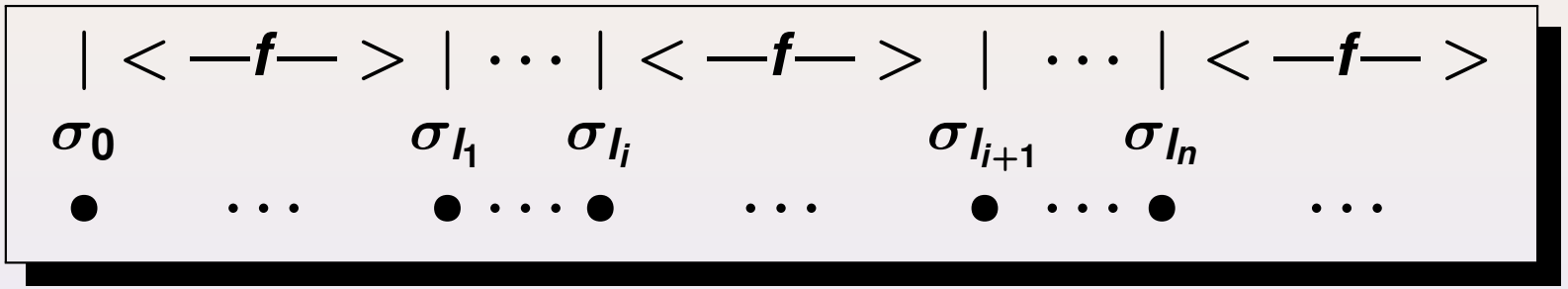
Finite interval σ is the fusion of a finite number of finite sub-intervals each satisfying f.

else

Semantics of formulae (87)

else

(exist $l_0, \ldots, l_n$ s.t. $l_0 = 0$ and
$[\![f]\!]_{\sigma_{l_n} \ldots |\sigma|} =$ tt and
for all $0 \leq i < n, l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} =$ tt)



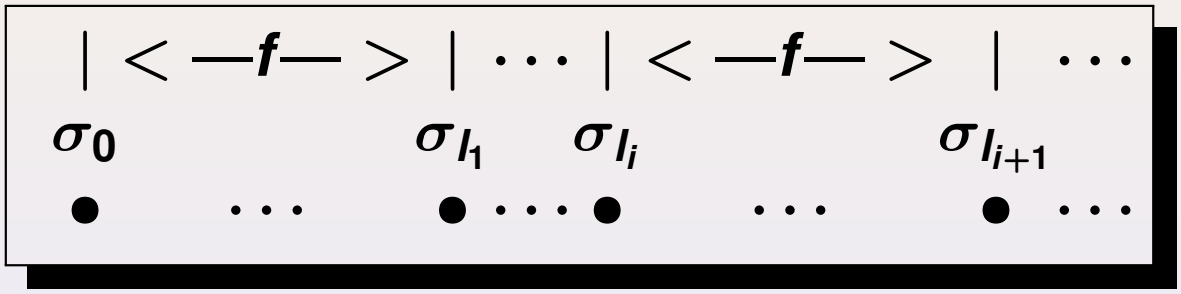
Infinite interval σ is the fusion of a finite number of sub-intervals each satisfying f. Each sub-interval is finite except the last one which is infinite.

or

Semantics of formulae

or

(exist an infinite number of l_i s.t. $l_0 = 0$ and
for all $0 \leq i, l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} = \mathrm{tt}$)



Infinite interval σ is the fusion of an infinite number of finite sub-intervals each satisfying f.

Derived temporal formulae (89)

Here are derived ITL formulae that use temporal variables:

$$
\begin{array}{lcll}
\boldsymbol{A := e} & \widehat{=} & (\bigcirc \boldsymbol{A}) = \boldsymbol{e} & \text{assignment} \\
\boldsymbol{A} \leftarrow \boldsymbol{e} & \widehat{=} & \text{finite} \wedge (\text{fin } \boldsymbol{A}) = \boldsymbol{e} & \text{temporal assignment} \\
\boldsymbol{A} \text{ gets } \boldsymbol{e} & \widehat{=} & \boxed{a}(\text{skip} \supset \boldsymbol{A} \leftarrow \boldsymbol{e}) & \text{gets} \\
\text{stable } \boldsymbol{A} & \widehat{=} & \boldsymbol{A} \text{ gets } \boldsymbol{A} & \text{stability} \\
\text{len}(\boldsymbol{n}) & \widehat{=} & \exists \boldsymbol{I} \bullet (\boldsymbol{I} = \mathbf{0}) \wedge (\boldsymbol{I} \text{ gets } \boldsymbol{I} + \mathbf{1}) & \\
 & & \quad \wedge (\boldsymbol{I} \leftarrow \boldsymbol{n}) & \text{interval length}
\end{array}
$$

Exercise

Exercise

Give the semantics of the following formulae:

true
false
$f_1 \vee f_2$
$f_1 \supset f_2$
$f_1 \equiv f_2$
$\exists v \bullet f$
if f_0 then f_1 else f_2

Exercise

Exercise

Give the formal semantics of $(\text{skip} \wedge \boldsymbol{Account} = \mathbf{50})^*$

Exercise

Exercise

Give the informal semantics (picture) of following formulae:

$\boldsymbol{A = 0} \wedge \text{skip} \wedge \boldsymbol{A := A + 1}$

$\text{len}\boldsymbol{(6)}$

$\boldsymbol{A = 0} \wedge \text{len}\boldsymbol{(2)} \wedge \boldsymbol{A \leftarrow A + 1}$

$\text{len}\boldsymbol{(4)} \wedge \boldsymbol{A = 3} \wedge \boldsymbol{A}\ \text{gets}\ \boldsymbol{A + 1}$

$\text{len}\boldsymbol{(3)} \wedge \boldsymbol{A = 2} \wedge \text{stable}\ \boldsymbol{A}$

Exercise

Exercise

Given informal specification

- *the system's behaviour consists of only two states,*
- *in the initial state the variable* ***Account*** *is equal to* **50** *and*
- *in the next state it is increased by* **100**.

Give the corresponding ITL formula.

Exercise

Exercise

Given the following "formal specification"

$$\begin{aligned}\textbf{\textit{Spec}} = \{\sigma : \quad & \sigma = \sigma_0 \ldots \sigma_n \textit{ and} \\ & \sigma_0(\textbf{\textit{Cash}}) = \textbf{\textit{initial}} \textit{ and} \\ & \sigma_{i+1}(\textbf{\textit{Cash}}) = \sigma_i(\textbf{\textit{Cash}}) + \mathbf{50} \\ & \textit{for } \mathbf{0} \leq \textbf{\textit{i}} < \textbf{\textit{n}} \\ \}\end{aligned}$$

Give the corresponding ITL formula.