Interval Temporal Algebra

(Automated Theorem Prover for PITL)

Antonio Cau

Website Interval Temporal Algebra slides

Overview (2)

This talk will introduce Interval Temporal Algebra

1. Introduction
2. Rationale
3. Propositional Interval Temporal Logic (PITL)
4. Algebraic semantics for PITL
5. Kleene and Omega Algebra
6. Interval Temporal Algebra
7. Automatic theorem prover for PITL
8. Conclusion and future work

Rationale (3)

An Automated Theorem Prover (ATP) for Interval Temporal Logic (ITL) has always been a desirable tool. There have been various attempts to implement such ATPs:

- Lite. Shinji Kono (1991 (Prolog), 2008 (Java)).
  Tableau-based, exponential in the number of the variables in the formula and also combinatorial w.r.t. to the nesting of temporal logic operators.
- ITL Library for PVS. Antonio Cau (1997).
  Interactive theorem prover for (in)finite first order ITL, requires expert proof knowledge and has little automation.
- DCVALID. Paritosh Pandya (2000).
  Quantified Discrete time Duration Calculus (QDDC). Translate QDDC into WS1S, use decision procedure of WS1S (MONA, Nils Klarlund and Anders Møller). WS1S has non-elementary complexity.

Rationale (4)

- PITL2Mona. Rodolfo Gomez (2004).
  Translate finite Propositional ITL (PITL) into WS1S, use decision procedure of WS1S (MONA). WS1S has non-elementary complexity.
- FL2CUDD. Ben Moszkowski (2005).
  Use a subset of (in)finite PITL called Fusion Logic (FL). The decision procedure of FL is built on top of Colorado University Decision Diagram (CUDD, Fabio Somenzi).

So mainly modelchecking or special purpose automated deduction.

There are off-the-shelf automated proof and counterexample search tools (ATP) for first-order and equational logic.

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Rationale (5)

Inspired by the work of Peter Höffner, Georg Struth and Bernhard Möller we proceed as follows:

- Introduce Interval Temporal Algebra (ITA). An ITA is based on Kleene Algebra and Omega Algebra.
- Show that PITL is an ITA.
- Encode the axiom system for ITA in an off-the shelf ATP Prover9 (first-order and equational logic, William McCune).

PITL (6)

Propositional Interval Temporal Logic (PITL) is a

- discrete,
- linear temporal logic
- for both finite and infinite time which includes
- a basic construct for sequential composition and
- an analog of Kleene star and Omega star

Syntax of PITL

$$f ::= \quad p \mid \neg f \mid f_1 \vee f_2 \mid \text{skip} \mid f_1 \,;\, f_2 \mid f^*$$

where

- skip is an interval (sequence) of 2 states.
- $f_1 \,;\, f_2$ is called 'f_1 chop f_2' and denotes sequential composition of two intervals.
- f^* is called 'f chopstar' and denotes (in)finite iteration of an interval.

Derived formulae

$\bigcirc\, \boldsymbol{f}$	$\widehat{=}$	$\mathsf{skip} \;;\; \boldsymbol{f}$	next
$\textcircled{w}\, \boldsymbol{f}$	$\widehat{=}$	$\neg \bigcirc \neg \boldsymbol{f}$	weak next
more	$\widehat{=}$	$\bigcirc \mathsf{true}$	interval with $\geq$ 2 states
empty	$\widehat{=}$	$\neg \mathsf{more}$	one state interval
inf	$\widehat{=}$	$\mathsf{true} \;;\; \mathsf{false}$	infinite interval
finite	$\widehat{=}$	$\neg \mathsf{inf}$	finite interval
fmore	$\widehat{=}$	$\mathsf{more} \wedge \mathsf{finite}$	finite with $\geq$ 2 states
$\Diamond\, \boldsymbol{f}$	$\widehat{=}$	$\mathsf{finite} \;;\; \boldsymbol{f}$	sometimes
$\Box\, \boldsymbol{f}$	$\widehat{=}$	$\neg \Diamond \neg \boldsymbol{f}$	always
$\Diamond\!\!\!\!\!\;{\scriptstyle i}\;\, \boldsymbol{f}$	$\widehat{=}$	$\boldsymbol{f} \;;\; \mathsf{true}$	some initial subinterval
$\boxed{i}\, \boldsymbol{f}$	$\widehat{=}$	$\neg(\Diamond\!\!\!\!\!\;{\scriptstyle i}\;\, \neg \boldsymbol{f})$	all initial subintervals
$\Diamond\!\!\!\!\!\;{\scriptstyle a}\;\, \boldsymbol{f}$	$\widehat{=}$	$\mathsf{finite} \;;\; \boldsymbol{f} \;;\; \mathsf{true}$	some subinterval
$\boxed{a}\, \boldsymbol{f}$	$\widehat{=}$	$\neg(\Diamond\!\!\!\!\!\;{\scriptstyle a}\;\, \neg \boldsymbol{f})$	all subintervals

etc.

Semantics of PITL (9)

The main semantic notion is interval which is a sequence of states.
Let

- $\mathbf{\Sigma}$ denotes the set of states.
- $\mathbf{\Sigma}^+$ denote the set of non-empty finite sequences of states.
- $\mathbf{\Sigma}^\omega$ denote the set of infinite sequences of states.
- σ denote an interval, $\sigma \in \mathbf{\Sigma}^+ \cup \mathbf{\Sigma}^\omega$.
- Let $[\![\ldots]\!]$ be the "meaning" (semantic) function from $\mathbf{\Sigma}^+ \cup \mathbf{\Sigma}^\omega$ to $\{\mathsf{tt}, \mathsf{ff}\}$.

Semantics of PITL (10)

- $[\![p]\!]_\sigma = \text{tt}$ iff $\sigma_0(p) = \text{tt}$
- $[\![\neg f]\!]_\sigma = \text{tt}$ iff not $([\![f]\!]_\sigma = \text{tt})$
- $[\![f_1 \vee f_2]\!]_\sigma = \text{tt}$ iff $[\![f_1]\!]_\sigma = \text{tt}$ or $[\![f_2]\!]_\sigma = \text{tt}$
- $[\![\text{skip}]\!]_\sigma = \text{tt}$ iff $|\sigma| = 1$ where $|\sigma|$ denotes length of σ and is defined as number of states minus 1
- $[\![f_1\ ;\ f_2]\!]_\sigma = \text{tt}$ iff
  (exists k, s.t. $[\![f_1]\!]_{\sigma_0 \ldots \sigma_k} = \text{tt}$ and $[\![f_2]\!]_{\sigma_k \ldots \sigma_{|\sigma|}} = \text{tt}$)
  or (σ is infinite and $[\![f_1]\!]_\sigma = \text{tt}$)

Semantics of PITL (11)

- $[\![f^*]\!]_\sigma =$ tt iff
  if σ is finite then
  (exist $l_0, \ldots, l_n$ s.t. $l_0 = 0$ and $l_n = |\sigma|$ and
  for all $0 \leq i < n$, $l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} =$ tt)
  else
  (exist $l_0, \ldots, l_n$ s.t. $l_0 = 0$ and
  $[\![f]\!]_{\sigma_{l_n} \ldots \sigma_{|\sigma|}} =$ tt and
  for all $0 \leq i < n$, $l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} =$ tt)
  or
  (exist an infinite number of l_i s.t. $l_0 = 0$ and
  for all $0 \leq i$, $l_i \leq l_{i+1}$ and $[\![f]\!]_{\sigma_{l_i} \ldots \sigma_{l_{i+1}}} =$ tt)

Axiom/proof system for PITL (12)

PropAx	All axioms for propositional logic
ChopAssoc	$\vdash\ (f_0 \,;\, f_1) \,;\, f_2 \ \equiv\ f_0 \,;\, (f_1 \,;\, f_2)$
OrChopImp	$\vdash\ (f_0 \vee f_1) \,;\, f_2 \ \supset\ (f_0 \,;\, f_2) \vee (f_1 \,;\, f_2)$
ChopOrImp	$\vdash\ f_0 \,;\, (f_1 \vee f_2) \ \supset\ (f_0 \,;\, f_1) \vee (f_0 \,;\, f_2)$
EmptyChop	$\vdash\ \mathrm{empty} \,;\, f_1 \ \equiv\ f_1$
ChopEmpty	$\vdash\ f_1 \,;\, \mathrm{empty} \ \equiv\ f_1$
BiBoxChop	$\vdash\ \boxed{\mathrm{i}}(f_0 \ \supset\ f_1) \wedge \Box(f_2 \ \supset\ f_3) \ \supset\ (f_0 \,;\, f_2) \ \supset\ (f_1 \,;\, f_3)$
StateImpBi	$\vdash\ p \ \supset\ \boxed{\mathrm{i}}\, p$
NextImpWNext	$\vdash\ \bigcirc f_0 \ \supset\ \neg \bigcirc \neg f_0$
SkipAnd	$\vdash\ (\mathrm{skip} \wedge f_0) \,;\, \mathrm{true} \ \supset\ \neg((\mathrm{skip} \wedge \neg f_0) \,;\, \mathrm{true})$
BoxInduct	$\vdash\ f_0 \wedge \Box(f_0 \ \supset\ \neg \bigcirc \neg f_0) \ \supset\ \Box\, f_0$
ChopStarEqv	$\vdash\ f_0^* \ \equiv\ (\mathrm{empty} \vee ((f_0 \wedge \mathrm{more}) \,;\, f_0^*))$
ChopStarInduct	$\vdash\ (\mathrm{inf} \wedge f_0 \wedge \Box(f_0 \ \supset\ (f_1 \wedge \mathrm{fmore}) \,;\, f_0)) \ \supset\ f_1^*$
MP	$\vdash\ f_0 \ \supset\ f_1$ and $\vdash\ f_0$ implies $\vdash\ f_1$
BoxGen	$\vdash\ f_0$ implies $\vdash\ \Box\, f_0$
BiGen	$\vdash\ f_0$ implies $\vdash\ \boxed{\mathrm{i}}\, f_0$

Algebraic semantics for PITL (13)

Can we give the semantic domain an algebraic structure?

Let $[\![f]\!]$ denote the set of intervals for which $[\![f]\!]_\sigma = \mathrm{tt}$, i.e.,

$$[\![f]\!] \mathrel{\widehat{=}} \{\sigma \mid [\![f]\!]_\sigma = \mathrm{tt}\}$$

The $\vee$ of two PITL formula is then

$[\![f_1 \vee f_2]\!] =$
— definition of $[\![\ ]\!]$
$\{\sigma \mid [\![f_1 \vee f_2]\!]_\sigma = \mathrm{tt}\}$
— definition of $[\![f_1 \vee f_2]\!]_\sigma$
$\{\sigma \mid [\![f_1]\!]_\sigma = \mathrm{tt}$ or $[\![f_2]\!]_\sigma = \mathrm{tt}\}$
— set theory, let $\cup$ denote union
$\{\sigma \mid [\![f_1]\!]_\sigma = \mathrm{tt}\} \cup \{\sigma \mid [\![f_2]\!]_\sigma = \mathrm{tt}\}$
— definition of $[\![\ ]\!]$
$[\![f_1]\!] \cup [\![f_2]\!]$

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— definition of $[\![\ ]\!]$
$\{\sigma \mid [\![f_1 \vee f_2]\!]_\sigma = \mathrm{tt}\}$
— definition of $[\![f_1 \vee f_2]\!]_\sigma$
$\{\sigma \mid [\![f_1]\!]_\sigma = \mathrm{tt} \text{ or } [\![f_2]\!]_\sigma = \mathrm{tt}\}$
— set theory, let $\cup$ denote union
$\{\sigma \mid [\![f_1]\!]_\sigma = \mathrm{tt}\} \cup \{\sigma \mid [\![f_2]\!]_\sigma = \mathrm{tt}\}$
— definition of $[\![\ ]\!]$
$[\![f_1]\!] \cup [\![f_2]\!]$

Algebraic semantics for PITL (14)

So we need algebraic operators that correspond to $\neg$, $\vee$, **skip**, **;** and *

- $\vee$ corresponds to union ($\cup$) of sets of intervals
- $\neg$ corresponds to complement, i.e.,

$$[\![\neg f]\!] =$$
— definition of $[\![\ ]\!]$
$$\{\sigma \mid [\![\neg f]\!]_\sigma = \mathrm{tt}\}$$
— definition of $[\![\neg f]\!]_\sigma$
$$\{\sigma \mid \mathrm{not}\ ([\![f]\!]_\sigma = \mathrm{tt})\}$$
— set theory, let $\overline{}$ denote set complement
$$\overline{\{\sigma \mid [\![f]\!]_\sigma = \mathrm{tt}\}}$$
— definition of $[\![\ ]\!]$
$$\overline{[\![f]\!]}$$

Algebraic semantics for PITL (15)

What about chop (‘;’)?

Let $\cdot$ denote the fusion of two intervals $\sigma_1, \sigma_2 \in \Sigma^+ \cup \Sigma^\omega$, i.e., Let $a, b \in \Sigma$ (a and b are not the same), $v, w \in \Sigma^*$ and $s, t \in \Sigma^\omega$

$$\sigma_1 \cdot \sigma_2 \mathrel{\widehat{=}} \begin{cases} vaw & \text{if} \quad \sigma_1 = va, \quad \sigma_2 = aw \\ \emptyset & \text{if} \quad \sigma_1 = va, \quad \sigma_2 = bw \\ vas & \text{if} \quad \sigma_1 = va, \quad \sigma_2 = as \\ \emptyset & \text{if} \quad \sigma_1 = va, \quad \sigma_2 = bs \\ s & \text{if} \quad \sigma_1 = s, \quad \sigma_2 = aw \\ s & \text{if} \quad \sigma_1 = s, \quad \sigma_2 = t \end{cases}$$

Let $S, T \subseteq \Sigma^+ \cup \Sigma^\omega$ then

$$S \cdot T \mathrel{\widehat{=}} \{\sigma_1 \cdot \sigma_2 | \sigma_1 \in S \text{ and } \sigma_2 \in T\}$$

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Algebraic semantics for PITL (16)

- '$;$' corresponds to fusion '$\cdot$', i.e.,

$$\llbracket f_1 \,;\, f_2 \rrbracket =$$
— definition of $\llbracket\ \rrbracket$
$$\{\sigma \mid \llbracket f_1 \,;\, f_2 \rrbracket_\sigma = \mathrm{tt}\}$$
— definition of $\llbracket f_1 \,;\, f_2 \rrbracket_\sigma$
$$\{\sigma \mid (\text{exists } k,\ \text{s.t. } \llbracket f_1 \rrbracket_{\sigma_0 \ldots \sigma_k} = \mathrm{tt} \text{ and } \llbracket f_2 \rrbracket_{\sigma_k \ldots \sigma_{|\sigma|}} = \mathrm{tt}) \text{ or } (\sigma \text{ is infinite and } \llbracket f_1 \rrbracket_\sigma = \mathrm{tt})\}$$
— definition of $\cdot$
$$\{\sigma \mid \llbracket f_1 \rrbracket_\sigma = \mathrm{tt}\} \cdot \{\sigma \mid \llbracket f_2 \rrbracket_\sigma = \mathrm{tt}\}$$
— definition of $\llbracket\ \rrbracket$
$$\llbracket f_1 \rrbracket \cdot \llbracket f_2 \rrbracket$$

Algebraic semantics for PITL (17)

What about empty?

$[\![\text{empty}]\!] =$
— definition of $[\![\ ]\!]$
$\{\sigma | [\![\text{empty}]\!]_\sigma = \text{tt}\}$
— definition of $[\![\text{empty}]\!]_\sigma$
$\{\sigma | \sigma \text{ is a } \mathbf{1} \text{ state interval}\}$
— definition of $\mathbf{\Sigma}$
$\mathbf{\Sigma}$

Algebraic semantics for PITL (17)

What about empty?

$[\![\text{empty}]\!] =$

— definition of $[\![\ ]\!]$

$\{\sigma | [\![\text{empty}]\!]_\sigma = \text{tt}\}$

— definition of $[\![\text{empty}]\!]_\sigma$

$\{\sigma | \sigma \text{ is a } \mathbf{1} \text{ state interval}\}$

— definition of $\mathbf{\Sigma}$

$\mathbf{\Sigma}$

What about skip?

skip can be defined as $\overline{\mathbf{\Sigma} \cup \overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}}$

$$\overline{\mathbf{\Sigma} \cup \overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}} =$$

— De Morgan for set theory

$$\overline{\mathbf{\Sigma}} \cap \overline{\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}}$$

$\overline{\mathbf{\Sigma}}$ is the set of intervals containing $\geq \mathbf{2}$ states
$\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}$ is the set of intervals containing $\geq \mathbf{3}$ states
$\overline{\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}}$ is the set of intervals containing $\leq \mathbf{2}$ states
$\overline{\mathbf{\Sigma}} \cap \overline{\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}}$ is the set of intervals containing exactly 2 states

Algebraic semantics for PITL (18)

What about skip?

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$$\overline{\mathbf{\Sigma} \cup \overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}} =$$

— De Morgan for set theory

$$\overline{\mathbf{\Sigma}} \cap \overline{\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}}$$

$\overline{\mathbf{\Sigma}}$ is the set of intervals containing $\geq$ **2** states

$\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}$ is the set of intervals containing $\geq$ **3** states

$\overline{\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}}$ is the set of intervals containing $\leq$ **2** states

$\overline{\mathbf{\Sigma}} \cap \overline{\overline{\mathbf{\Sigma}} \cdot \overline{\mathbf{\Sigma}}}$ is the set of intervals containing exactly 2 states

Algebraic semantics for PITL (19)

What about a state formula, i.e., a formula without temporal operators?

A state formula only constrains the first state of an interval. Let p be a state formula. Then the following holds

$$[\![p]\!] = ([\![p]\!] \cap \Sigma) \cdot \mathsf{T}$$

where

$$\mathsf{T} \mathrel{\widehat{=}} [\![\mathrm{true}]\!] = \Sigma^+ \cup \Sigma^\omega$$
$$\emptyset \mathrel{\widehat{=}} [\![\mathrm{false}]\!] = \emptyset$$

Algebraic semantics for PITL

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where

$$\begin{aligned} \mathsf{T} &\mathrel{\widehat{=}} [\![\mathsf{true}]\!] = \boldsymbol{\Sigma}^{+} \cup \boldsymbol{\Sigma}^{\omega} \\ \emptyset &\mathrel{\widehat{=}} [\![\mathsf{false}]\!] = \emptyset \end{aligned}$$

Algebraic semantics for PITL (20)

What about chopstar '$*$'?

In the semantics of '$*$' both finite and infinite iteration are considered simultaneously. Let's define separate algebraic operators for them.

Let S^* and S^ω denote respectively finite and infinite iteration of a set $S \subseteq \Sigma^+ \cup \Sigma^\omega$ and can be defined as follows

$$
\begin{array}{lcl|lcl}
f(X) & \widehat{=} & \Sigma \cup S \cdot X & g(X) & \widehat{=} & S \cdot X \\
f^0(X) & \widehat{=} & X & g^0(X) & \widehat{=} & X \\
f^{i+1}(X) & \widehat{=} & f(f^i(X)) & g^{i+1}(X) & \widehat{=} & g(g^i(X)) \\
S^* & \widehat{=} & \bigcup_i f^i(\emptyset) & S^\omega & \widehat{=} & \bigcap_i g^i(\top)
\end{array}
$$

Then we have

$$[\![f^*]\!] = \ldots = [\![f]\!]^* \cup [\![f]\!]^\omega$$

Algebraic semantics for PITL

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Kleene and Omega Algebra (21)

Let K denote $\mathcal{P}(\Sigma^+ \cup \Sigma^\omega)$,
$(K, \cup, \emptyset, \cdot, \Sigma)$ is an idempotent left semiring iff
(for $a, b, c \in K$)

- $(K, \cup, \emptyset)$ is a commutative monoid , i.e.,
  $a \cup b = b \cup a$
  $a \cup \emptyset = a$
  $a \cup (b \cup c) = (a \cup b) \cup c$
- $(K, \cdot, \Sigma)$ is a monoid, i.e.,
  $a \cdot \Sigma = a$
  $\Sigma \cdot a = a$
  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- $(a \cup b) \cdot c = a \cdot c \cup b \cdot c$
- $\emptyset \cdot a = \emptyset$
- $a \cup a = a$
- $b \subseteq c$ implies $a \cdot b \subseteq a \cdot c$ where $a \subseteq b$ iff $a \cup b = b$

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Kleene and Omega Algebra

$(K, \overline{})$ is a Boolean idempotent left semiring iff

- $(K, \cup, \emptyset, \cdot, \Sigma)$ is an idempotent left semiring
- $a = \overline{(\overline{a} \cup \overline{b})} \cup \overline{(\overline{a} \cup b)}$ (Huntington equation)

As usual we have the following

- $a \cap b = \overline{\overline{a} \cup \overline{b}}$
- $\top = a \cup \overline{a}$ (greatest element w.r.t. $\subseteq$, i.e. $\Sigma^+ \cup \Sigma^\omega$)
- $\emptyset = a \cap \overline{a}$ (smallest element w.r.t. $\subseteq$, i.e., $\emptyset$)

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- $a \cap b = \overline{\overline{a} \cup \overline{b}}$
- $\top = a \cup \overline{a}$ (greatest element w.r.t. $\subseteq$, i.e. $\Sigma^+ \cup \Sigma^\omega$)
- $\emptyset = a \cap \overline{a}$ (smallest element w.r.t. $\subseteq$, i.e., $\emptyset$)

Kleene and Omega Algebra (23)

$(K, {}^*)$ is a Boolean strong left (lazy) Kleene algebra iff

- $(K, \overline{\ })$ is a Boolean idempotent left semiring
- $\Sigma \cup a \cdot a^* \subseteq a^*$
- $b \cup a \cdot c \subseteq c$ implies $a^* \cdot b \subseteq c$
- $b \cup c \cdot a \subseteq c$ implies $b \cdot a^* \subseteq c$

$(K, {}^\omega)$ is a Boolean left (lazy) omega algebra iff

- $(K, {}^*)$ is a Boolean strong left (lazy) Kleene algebra
- $a^\omega = a \cdot a^\omega$
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Interval Temporal Algebra (24)

$(K, {}^\omega)$ is an interval temporal algebra iff

Let $\mathsf{skip} \mathrel{\widehat{=}} \overline{\Sigma \cup \overline{\Sigma} \cdot \overline{\Sigma}}$, $\mathsf{finite} \mathrel{\widehat{=}} \overline{\top \cdot \emptyset}$ and $\Box a \mathrel{\widehat{=}} \overline{\mathsf{finite} \cdot \overline{a}}$

P1 $(K, {}^\omega)$ is a Boolean left (lazy) omega algebra

P2 $a \cdot (b \cup c) \subseteq a \cdot b \cup a \cdot c$

P3 $(\mathsf{skip} \cap a) \cdot c \cap (\mathsf{skip} \cap b) \cdot d = (\mathsf{skip} \cap a \cap b) \cdot (c \cap d)$

P4 $c \cdot (\mathsf{skip} \cap a) \cap d \cdot (\mathsf{skip} \cap b) = (c \cap d) \cdot (\mathsf{skip} \cap a \cap b)$

P5 $(\Sigma \cap a) \cdot c \cap (\Sigma \cap b) \cdot d = (\Sigma \cap a \cap b) \cdot (c \cap d)$

P6 $c \cdot (\Sigma \cap a) \cap d \cdot (\Sigma \cap b) = (c \cap d) \cdot (\Sigma \cap a \cap b)$

P7 $\mathsf{finite} \subseteq \mathsf{skip}^*$

P8 $\overline{\mathsf{skip} \cdot \overline{a}} = \Sigma \cup \mathsf{skip} \cdot a$

P9 $\overline{\overline{a} \cdot \mathsf{skip}} = \Sigma \cup a \cdot \mathsf{skip}$

Axiom/proof system for PITL (25)

PropAx	All axioms for propositional logic
ChopAssoc	$\vdash\ (f_0 ; f_1) ; f_2 \ \equiv\ f_0 ; (f_1 ; f_2)$
OrChopImp	$\vdash\ (f_0 \vee f_1) ; f_2 \ \supset\ (f_0 ; f_2) \vee (f_1 ; f_2)$
ChopOrImp	$\vdash\ f_0 ; (f_1 \vee f_2) \ \supset\ (f_0 ; f_1) \vee (f_0 ; f_2)$
EmptyChop	$\vdash\ \mathrm{empty} ; f_1 \ \equiv\ f_1$
ChopEmpty	$\vdash\ f_1 ; \mathrm{empty} \ \equiv\ f_1$
BiBoxChop	$\vdash\ \boxdot(f_0 \supset f_1) \wedge \Box(f_2 \supset f_3) \ \supset\ (f_0 ; f_2) \ \supset\ (f_1 ; f_3)$
StateImpBi	$\vdash\ p \ \supset\ \boxdot\, p$
NextImpWNext	$\vdash\ \bigcirc f_0 \ \supset\ \neg\bigcirc\neg f_0$
SkipAnd	$\vdash\ (\mathrm{skip} \wedge f_0) ; \mathrm{true} \ \supset\ \neg((\mathrm{skip} \wedge \neg f_0) ; \mathrm{true})$
BoxInduct	$\vdash\ f_0 \wedge \Box(f_0 \supset \neg\bigcirc\neg f_0) \ \supset\ \Box\, f_0$
ChopStarEqv	$\vdash\ f_0^* \ \equiv\ (\mathrm{empty} \vee ((f_0 \wedge \mathrm{more}) ; f_0^*))$
ChopStarInduct	$\vdash\ (\mathrm{inf} \wedge f_0 \wedge \Box(f_0 \supset (f_1 \wedge \mathrm{fmore}) ; f_0)) \ \supset\ f_1^*$
MP	$\vdash\ f_0 \supset f_1$ and $\vdash\ f_0$ implies $\vdash\ f_1$
BoxGen	$\vdash\ f_0$ implies $\vdash\ \Box\, f_0$
BiGen	$\vdash\ f_0$ implies $\vdash\ \boxdot\, f_0$

Examples of proof in ITA (26)

Example

$$
\begin{aligned}
& \mathsf{finite} \cdot \boldsymbol{\Sigma} \\
& \quad — \ \boldsymbol{a} \cdot \boldsymbol{\Sigma} = \boldsymbol{a} \\
= \ & \mathsf{finite}
\end{aligned}
$$

Example

$$
\begin{aligned}
& (\mathsf{T} \cdot \emptyset) \cdot \emptyset \\
& \quad — \ \boldsymbol{a} \cdot (\boldsymbol{b} \cdot \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b}) \cdot \boldsymbol{c} \\
= \ & \mathsf{T} \cdot (\emptyset \cdot \emptyset) \\
& \quad — \ \emptyset \cdot \boldsymbol{a} = \emptyset \\
= \ & \mathsf{T} \cdot \emptyset
\end{aligned}
$$

ITA vs PITL

Theorem

PITL is an interval temporal algebra

Theorem

PITL's axiom/proof system can be derived from ITA's axiom system

Automatic theorem prover for PITL (28)

- Take an off-the-shelf automatic theorem prover (ATP)
- encode the algebraic rules for an interval temporal algebra within it

We used Prover9 as ATP and encoded the interval temporal algebraic rules.

- Proved more than 350 PITL theorems so far
- Derived all PITL axioms/proof rules from ITA's axioms

Demo

Conclusion and future work

- An interval temporal algebraic encoding of PITL in Prover9 results in quite a useful tool
- need to reduce the axioms of ITA
- replace some of the more complex axioms of ITA by simpler ones,
- Use a different ATP
- Integration of Description Logic axiom system with Interval Temporal Algebra axiom system